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Differential Geometry via Moving Frames and Exterior Differential Systems

微分几何中嘉当的活动标架法和外微分系统初步

Thomas A. Ivey, J. M. Landsberg

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

In this book, we use moving frames and exterior differential systems to study geometry and partial differential equations. These ideas originated about a century ago in the works of several mathematicians, including Gaston Darboux, Edouard Goursat and, most importantly, Elie Cartan. Over the years these techniques have been refined and extended; major contributors to the subject are mentioned below, under "Further Reading".

The book has the following features: It concisely covers the classical geometry of surfaces and basic Riemannian geometry in the language of moving frames. It includes results from projective differential geometry that update and expand the classic paper [**69**] of Griffiths and Harris. It provides an elementary introduction to the machinery of exterior differential systems (EDS), and an introduction to the basics of G-structures and the general theory of connections. Classical and recent geometric applications of these techniques are discussed throughout the text.

This book is intended to be used as a textbook for a graduate-level course; there are numerous exercises throughout. It is suitable for a one-year course, although it has more material than can be covered in a year, and parts of it are suitable for one-semester course (see the end of this preface for some suggestions). The intended audience is both graduate students who have some familiarity with classical differential geometry and differentiable manifolds, and experts in areas such as PDE and algebraic geometry who want to learn how moving frame and EDS techniques apply to their fields.

In addition to the geometric applications presented here, EDS techniques are also applied in CR geometry (see, e.g., [**98**]), robotics, and control theory (see [**55, 56, 129**]). This book prepares the reader for such areas, as well as

for more advanced texts on exterior differential systems, such as [**20**], and papers on recent advances in the theory, such as [**58, 117**].

Overview. Each section begins with geometric examples and problems. Techniques and definitions are introduced when they become useful to help solve the geometric questions under discussion. We generally keep the presentation elementary, although advanced topics are interspersed throughout the text.

In Chapter 1, we introduce moving frames via the geometry of curves in the Euclidean plane $\mathbb{E}^2$. We define the Maurer-Cartan form of a Lie group G and explain its use in the study of submanifolds of G-homogeneous spaces. We give additional examples, including the equivalence of holomorphic mappings up to fractional linear transformation, where the machinery leads one naturally to the Schwarzian derivative.

We define exterior differential systems and jet spaces, and explain how to rephrase any system of partial differential equations as an EDS using jets. We state and prove the Frobenius system, leading up to it via an elementary example of an overdetermined system of PDE.

In Chapter 2, we cover traditional material—the geometry of surfaces in three-dimensional Euclidean space, submanifolds of higher-dimensional Euclidean space, and the rudiments of Riemannian geometry—all using moving frames. Our emphasis is on local geometry, although we include standard global theorems such as the rigidity of the sphere and the Gauss-Bonnet Theorem. Our presentation emphasizes finding and interpreting differential invariants to enable the reader to use the same techniques in other settings.

We begin Chapter 3 with a discussion of Grassmannians and the Plücker embedding. We present some well-known material (e.g., Fubini's theorem on the rigidity of the quadric) which is not readily available in other textbooks. We present several recent results, including the Zak and Landman theorems on the dual defect, and results of the second author on complete intersections, osculating hypersurfaces, uniruled varieties and varieties covered by lines. We keep the use of terminology and results from algebraic geometry to a minimum, but we believe we have included enough so that algebraic geometers will find this chapter useful.

Chapter 4 begins our multi-chapter discussion of the Cartan algorithm and Cartan-Kähler Theorem. In this chapter we study constant coefficient homogeneous systems of PDE and the linear algebra associated to the corresponding exterior differential systems. We define tableaux and involutivity of tableaux. One way to understand the Cartan-Kähler Theorem is as follows: given a system of PDE, if the linear algebra at the infinitesimal level

"works out right" (in a way explained precisely in the chapter), then existence of solutions follows.

In Chapter 5 we present the Cartan algorithm for linear Pfaffian systems, a very large class of exterior differential systems that includes systems of PDE rephrased as exterior differential systems. We give numerous examples, including many from Cartan's classic treatise [**31**], as well as the isometric immersion problem, problems related to calibrated submanifolds, and an example motivated by variation of Hodge structure.

In Chapter 6 we take a detour to discuss the classical theory of characteristics, Darboux's method for solving PDE, and Monge-Ampère equations in modern language. By studying the exterior differential systems associated to such equations, we recover the sine-Gordon representation of pseudospherical surfaces, the Weierstrass representation of minimal surfaces, and the one-parameter family of non-congruent isometric deformations of a surface of constant mean curvature. We also discuss integrable extensions and Bäcklund transformations of exterior differential systems, and the relationship between such transformations and Darboux integrability.

In Chapter 7, we present the general version of the Cartan-Kähler Theorem. Doing so involves a detailed study of the integral elements of an EDS. In particular, we arrive at the notion of a Kähler-regular flag of integral elements, which may be understood as the analogue of a sequence of well-posed Cauchy problems. After proving both the Cartan-Kähler Theorem and Cartan's test for regularity, we apply them to several examples of non-Pfaffian systems arising in submanifold geometry.

Finally, in Chapter 8 we give an introduction to geometric structures (G-structures) and connections. We arrive at these notions at a leisurely pace, in order to develop the intuition as to why one needs them. Rather than attempt to describe the theory in complete generality, we present one extended example, path geometry in the plane, to give the reader an idea of the general theory. We conclude with a discussion of some recent generalizations of G-structures and their applications.

There are four appendices, covering background material for the main part of the book: linear algebra and rudiments of representation theory, differential forms and vector fields, complex and almost complex manifolds, and a brief discussion of initial value problems and the Cauchy-Kowalevski Theorem, of which the Cartan-Kähler Theorem is a generalization.

Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in section 1.3. Equations are numbered sequentially within each chapter. We have included hints for selected exercises, those marked with the symbol ◎ at the end, which is meant to be suggestive of a life preserver.

Further Reading on EDS. To our knowledge, there are only a small number of textbooks on exterior differential systems. The first is Cartan's classic text [**31**], which has an extraordinarily beautiful collection of examples, some of which are reproduced here. We learned the subject from our teacher Bryant and the book by Bryant, Chern, Griffiths, Gardner and Goldschmidt [**20**], which is an elaboration of an earlier monograph [**19**], and is at a more advanced level than this book. One text at a comparable level to this book, but more formal in approach, is [**156**]. The monograph [**70**], which is centered around the isometric embedding problem, is similar in spirit but covers less material. The memoir [**155**] is dedicated to extending the Cartan-Kähler Theorem to the C^∞ setting for hyperbolic systems, but contains an exposition of the general theory. There is also a monograph by Kähler [**89**] and lectures by Kuranishi [**97**], as well the survey articles [**66, 90**]. Some discussion of the theory may be found in the differential geometry texts [**142**] and [**145**].

We give references for other topics discussed in the book in the text.

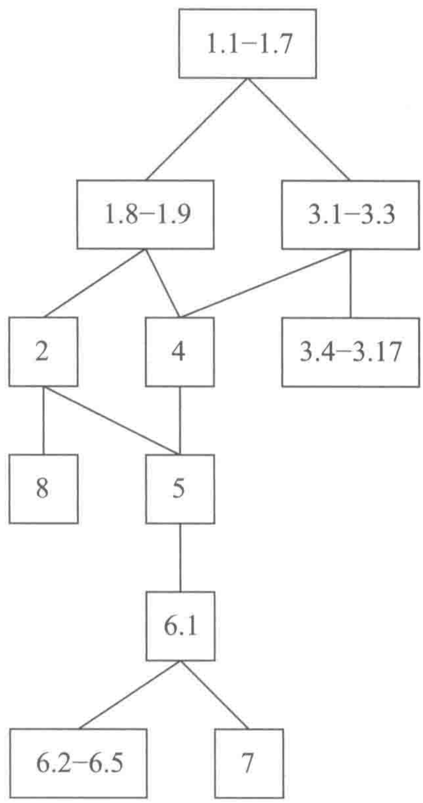
History and Acknowledgements. This book started out about a decade ago. We thought we would write up notes from Robert Bryant's Tuesday night seminar, held in 1988–89 while we were graduate students, as well as some notes on exterior differential systems which would be more introductory than [**20**]. The seminar material is contained in §8.6 and parts of Chapter 6. Chapter 2 is influenced by the many standard texts on the subject, especially [**43**] and [**142**], while Chapter 3 is influenced by the paper [**69**]. Several examples in Chapter 5 and Chapter 7 are from [**31**], and the examples of Darboux's method in Chapter 6 are from [**63**]. In each case, specific attributions are given in the text. Chapter 7 follows Chapter III of [**20**] with some variations. In particular, to our knowledge, Lemmas 7.1.10 and 7.1.13 are original. The presentation in §8.5 is influenced by [**11**], [**94**] and unpublished lectures of Bryant.

The first author has given graduate courses based on the material in Chapters 6 and 7 at the University of California, San Diego and at Case Western Reserve University. The second author has given year-long graduate courses using Chapters 1, 2, 4, 5, and 8 at the University of Pennsylvania and Université de Toulouse III, and a one-semester course based on Chapters 1, 2, 4 and 5 at Columbia University. He has also taught one-semester

undergraduate courses using Chapters 1 and 2 and the discussion of connections in Chapter 8 (supplemented by [**141**] and [**142**] for background material) at Toulouse and at Georgia Institute of Technology, as well as one-semester graduate courses on projective geometry from Chapters 1 and 3 (supplemented by some material from algebraic geometry), at Toulouse, Georgia Tech. and the University of Trieste. He also gave more advanced lectures based on Chapter 3 at Seoul National University, which were published as [**107**] and became a precursor to Chapter 3. Preliminary versions of Chapters 5 and 8 respectively appeared in [**104, 103**].

We would like to thank the students in the above classes for their feedback. We also thank Megan Dillon, Phillipe Eyssidieux, Daniel Fox, Sung-Eun Koh, Emilia Mezzetti, Joseph Montgomery, Giorgio Ottaviani, Jens Piontkowski, Margaret Symington, Magdalena Toda, Sung-Ho Wang and Peter Vassiliou for comments on the earlier drafts of this book, and Annette Rohrs for help with the figures. The staff of the publications division of the AMS—in particular, Ralph Sizer, Tom Kacvinsky, and our editor, Ed Dunne—were of tremendous help in pulling the book together. We are grateful to our teacher Robert Bryant for introducing us to the subject. Lastly, this project would not have been possible without the support and patience of our families.

Dependence of Chapters



Suggested uses of this book:

- a year-long graduate course covering moving frames and exterior differential systems (chapters 1–8);
- a one-semester course on exterior differential systems and applications to partial differential equations (chapters 1 and 4–7);
- a one-semester course on the use of moving frames in algebraic geometry (chapter 3, preceded by part of chapter 1);
- a one-semester beginning graduate course on differential geometry (chapters 1, 2 and 8).

Contents

Chapter 1

Moving Frames and Exterior Differential Systems

In this chapter we motivate the use of differential forms to study problems in geometry and partial differential equations. We begin with familiar material: the Gauss and mean curvature of surfaces in $\mathbb{E}^3$ in §1.1, and Picard's Theorem for local existence of solutions of ordinary differential equations in §1.2. We continue in §1.2 with a discussion of a simple system of partial differential equations, and then in §1.3 rephrase it in terms of differential forms, which facilitates interpreting it geometrically. We also state the Frobenius Theorem.

In §1.4, we review curves in $\mathbb{E}^2$ in the language of moving frames. We generalize this example in §§1.5–1.6, describing how one studies submanifolds of homogeneous spaces using moving frames, and introducing the Maurer-Cartan form. We give two examples of the geometry of curves in homogeneous spaces: classifying holomorphic mappings of the complex plane under fractional linear transformations in §1.7, and classifying curves in $\mathbb{E}^3$ under Euclidean motions (i.e., rotations and translations) in §1.8. We also include exercises on plane curves in other geometries.

In §1.9, we define exterior differential systems and integral manifolds. We prove the Frobenius Theorem, give a few basic examples of exterior differential systems, and explain how to express a system of partial differential equations as an exterior differential system using jet bundles.

Throughout this book we use the summation convention: unless otherwise indicated, summation is implied whenever repeated indices occur up and down in an expression.

1.1. Geometry of surfaces in $\mathbb{E}^3$ in coordinates

Let $\mathbb{E}^3$ denote Euclidean three-space, i.e., the affine space $\mathbb{R}^3$ equipped with its standard inner product.

Given two smooth surfaces $S, S' \subset \mathbb{E}^3$, when are they "equivalent"? For the moment, we will say that two surfaces are (locally) equivalent if there exist a rotation and translation taking (an open subset of) S onto (an open subset of) S'.

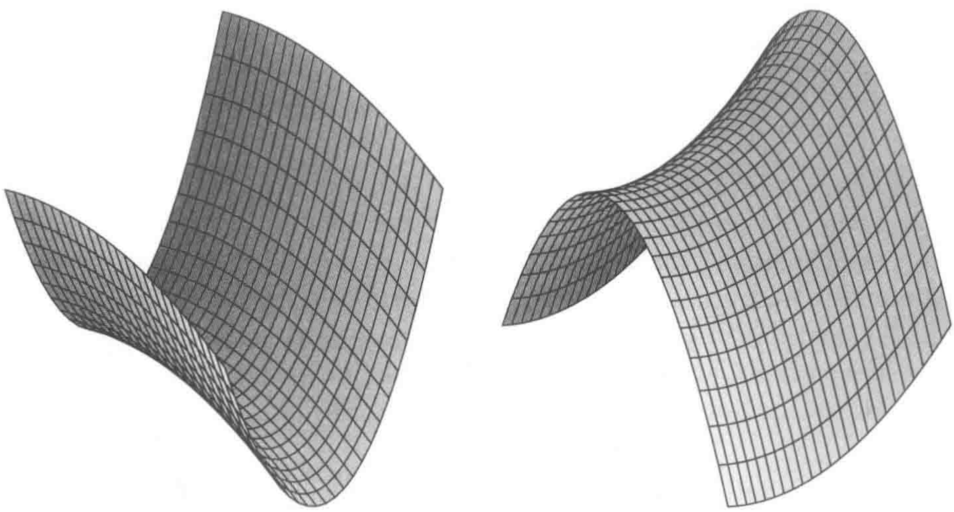


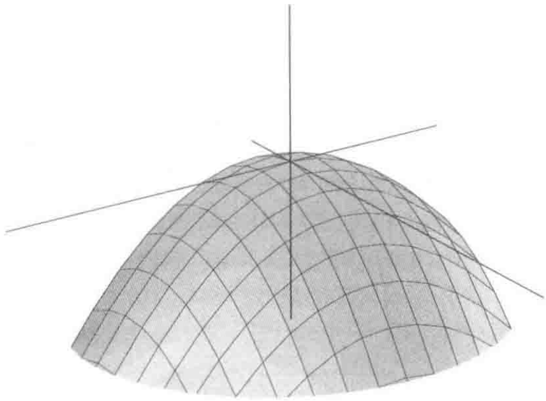
Figure 1. Are these two surfaces equivalent?

It would be impractical and not illuminating to try to test all possible motions to see if one of them maps S onto S'. Instead, we will work as follows:

Fix one surface S and a point $p \in S$. We will use the Euclidean motions to put S into a normalized position in space with respect to p. Then any other surface S' will be locally equivalent to S at p if there is a point $p' \in S'$ such that the pair (S', p') can be put into the same normalized position as (S, p).

The implicit function theorem implies that there always exist coordinates such that S is given locally by a graph $z = f(x, y)$. To obtain a normalized position for our surface S, first translate so that $p = (0,0,0)$, then use a rotation to make T_pS the xy-plane, i.e., so that $z_x(0,0) = z_y(0,0) = 0$. We

will call such coordinates *adapted to* p. At this point we have used up all our freedom of motion except for a rotation in the xy-plane.



If coordinates are adapted to p and we expand $f(x, y)$ in a Taylor series centered at the origin, then functions of the coefficients of the series that are invariant under this rotation are differential invariants.

In this context, a (Euclidean) *differential invariant* of S at p is a function I of the coefficients of the Taylor series for f at p, with the property that, if we perform a Euclidean change of coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is a rotation matrix and a, b, c are arbitrary constants, after which S is expressed as a graph $\tilde{z} = \tilde{f}(\tilde{x}, \tilde{y})$ near p, then I has the same value when computed using the Taylor coefficients of $\tilde{f}$ at p. Clearly a necessary condition for (S, p) to be locally equivalent to (S', p') is that the values of differential invariants of S at p match the values of the corresponding invariants of S' at p'.

For example, consider the Hessian of $z = z(x, y)$ at p:

$$\text{(1.1)} \qquad \mathrm{Hess}_p = \begin{pmatrix} z_{xx} & z_{yx} \\ z_{xy} & z_{yy} \end{pmatrix}\Bigg|_p .$$

Assume we are have adapted coordinates to p. If we rotate in the xy plane, the Hessian gets conjugated by the rotation matrix. The quantities

$$\text{(1.2)} \qquad \begin{aligned} K_0 &= \det(\mathrm{Hess}_p) = (z_{xx}z_{yy} - z_{xy}^2)\mid_p, \\ H_0 &= \tfrac{1}{2}\mathrm{trace}(\mathrm{Hess}_p) = \tfrac{1}{2}(z_{xx} + z_{yy})\mid_p . \end{aligned}$$

are differential invariants because the determinant and trace of a matrix are unchanged by conjugation by a rotation matrix. Thus, if we are given two surfaces S, S' and we normalize them both at respective points p and p' as

above, a necessary condition for there to be a rigid motion taking p' to p such that the Taylor expansions for the two surfaces at the point p coincide is that $K_0(S) = K_0(S')$ and $H_0(S) = H_0(S')$.

The formulas (1.2) are only valid at one point, and only after the surface has been put in normalized position relative to that point. To calculate K and H as functions on S it would be too much work to move each point to the origin and arrange its tangent plane to be horizontal. But it is possible to adjust the formulas to account for tilted tangent planes (see §2.10). One then obtains the following functions, which are differential invariants under Euclidean motions of surfaces that are locally described as graphs $z = z(x, y)$:

$$\begin{aligned} K(x,y) &= \frac{z_{xx}z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2}, \\ H(x,y) &= \frac{1}{2}\frac{(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}, \end{aligned} \tag{1.3}$$

respectively giving the *Gauss* and *mean* curvature of S at $p = (x, y, z(x, y))$.

Exercise 1.1.1: By locally describing each surface as a graph, calculate the Gauss and mean curvature functions for a sphere of radius R, a cylinder of radius r (e.g., $x^2 + y^2 = r^2$) and the smooth points of the cone $x^2 + y^2 = z^2$.

Once one has found invariants for a given submanifold geometry, one may ask questions about submanifolds with special invariants. For surfaces in $\mathbb{E}^3$, one might ask which surfaces have K constant or H constant. These can be treated as questions about solutions to certain partial differential equations (PDE). For example, from (1.3) we see that surfaces with $K \equiv 1$ are locally given by solutions to the PDE

$$z_{xx}z_{yy} - z_{xy}^2 = (1 + z_x^2 + z_y^2)^2. \tag{1.4}$$

We will soon free ourselves of coordinates and use moving frames and differential forms. As a provisional definition, a *moving frame* is a smoothly varying basis of the tangent space to $\mathbb{E}^3$ defined at each point of our surface. In general, using moving frames one can obtain formulas valid at every point analogous to coordinate formulas valid at just one preferred point. In the present context, the Gauss and mean curvatures will be described at all points by expressions like (1.2) rather than (1.3); see §2.1.

Another reason to use moving frames is that the method gives a uniform procedure for dealing with diverse geometric settings. Even if one is originally only interested in Euclidean geometry, other geometries arise naturally. For example, consider the *warp* of a surface, which is defined to be $(k_1 - k_2)^2$, where the k_j are the eigenvalues of (1.1). It turns out that this

quantity is invariant under a larger change of coordinates than the Euclidean group, namely *conformal* changes of coordinates, and thus it is easier to study the warp in the context of conformal geometry.

Regardless of how unfamiliar a geometry initially appears, the method of moving frames provides an algorithm to find differential invariants. Thus we will have a single method for dealing with conformal, Hermitian, projective and other geometries. Because it is familiar, we will often use the geometry of surfaces in $\mathbb{E}^3$ as an example, but the reader should keep in mind that the beauty of the method is its wide range of applicability. As for the use of differential forms, we shall see that when we express a system of PDE as an exterior differential system, the geometric features of the system—i.e., those which are independent of coordinates—will become transparent.

1.2. Differential equations in coordinates

The first questions one might ask when confronted with a system of differential equations are: Are there any solutions? If so, how many?

In the case of a single ordinary differential equation (ODE), here is the answer:

Theorem 1.2.1 (Picard[1]). *Let $f(x,u):\mathbb{R}^2\to\mathbb{R}$ be a function with f and f_u continuous. Then for all $(x_0,u_0)\in\mathbb{R}^2$, there exist an open interval $I\ni x_0$ and a function $u(x)$ defined on I, satisfying $u(x_0)=u_0$ and the differential equation*

$$\frac{du}{dx}=f(x,u). \tag{1.5}$$

Moreover, any other solution of this initial value problem must coincide with this solution on I.

In other words, for a given ODE there exists a solution defined near x_0 and this solution is unique given the choice of a constant u_0. Thus for an ODE for one function of one variable, we say that *solutions depend on one constant*. More generally, Picard's Theorem applies to systems of n first-order ODE's involving n unknowns, where solutions depend on n constants.

The graph in $\mathbb{R}^2$ of any solution to (1.5) is tangent at each point to the vector field $X=\frac{\partial}{\partial x}+f(x,u)\frac{\partial}{\partial u}$. This indicates how determined ODE systems generalize to the setting of differentiable manifolds (see Appendix B). If M is a manifold and X is a vector field on M, then a solution to the *system defined by* X is an immersed curve $c:I\to M$ such that $c'(t)=X_{c(t)}$ for all $t\in I$. (This is also referred to as an *integral curve* of X.) Away from singular points, one is guaranteed existence of local solutions to such systems and can even take the solution curves as coordinate curves:

[1]See, e.g., [**140**], p.423

Theorem 1.2.2 (Flowbox coordinates[2]). *Let M be an m-dimensional C^∞ manifold, let $p \in M$, and let $X \in \Gamma(TM)$ be a smooth vector field which is nonzero at p. Then there exists a local coordinate system $(x^1, \dots, x^m)$, defined in a neighborhood U of p, such that $\frac{\partial}{\partial x^1} = X$.*

Consequently, there exists an open set $V \subset U \times \mathbb{R}$ on which we may define the *flow of* X, $\phi : V \to M$, by requiring that for any point $q \in U$, $\frac{\partial}{\partial t}\phi(q,t) = X|_{\phi(q,t)}$ The flow is given in flowbox coordinates by

$$(x^1, \dots, x^m, t) \mapsto (x^1 + t, x^2, \dots, x^m).$$

With systems of PDE, it becomes difficult to determine the appropriate initial data for a given system (see Appendix D for examples). We now examine a simple PDE system, first in coordinates, and then later (in §5.2) using differential forms.

Example 1.2.3. Consider the system for $u(x,y)$ given by

$$\begin{aligned} u_x &= A(x,y,u), \\ u_y &= B(x,y,u), \end{aligned} \tag{1.6}$$

where A, B are given smooth functions. Since (1.6) specifies both partial derivatives of u, at any given point $p = (x,y,u) \in \mathbb{R}^3$ the tangent plane to the graph of a solution passing through p is uniquely determined.

In this way, (1.6) defines a smoothly-varying field of two-planes on $\mathbb{R}^3$, just as the ODE (1.5) defines a field of one-planes (i.e., a line field) on $\mathbb{R}^2$. For (1.5), Picard's Theorem guarantees that the one-planes "fit together" to form a solution curve through any given point. For (1.6), existence of solutions amounts to whether or not the two-planes "fit together".

We can attempt to solve (1.6) in a neighborhood of $(0,0)$ by solving a succession of ODE's. Namely, if we set $y = 0$ and $u(0,0) = u_0$, Picard's Theorem implies that there exists a unique function $\tilde{u}(x)$ satisfying

$$\frac{d\tilde{u}}{dx} = A(x, 0, \tilde{u}), \qquad \tilde{u}(0) = u_0. \tag{1.7}$$

After solving (1.7), hold x fixed and use Picard's Theorem again on the initial value problem

$$\frac{du}{dy} = B(x,y,u), \qquad u(x,0) = \tilde{u}(x). \tag{1.8}$$

This determines a function $u(x,y)$ on some neighborhood of $(0,0)$. The problem is that this function may not satisfy our original equation.

Whether or not (1.8) actually gives a solution to (1.6) depends on whether or not the equations (1.6) are "compatible" as differential equations. For smooth solutions to a system of PDE, compatibility conditions

[2]See, e.g., [**142**] vol. I, p.205

arise because mixed partials must commute, i.e., $(u_x)_y = (u_y)_x$. In our example,

$$\begin{aligned}(u_x)_y &= \frac{\partial}{\partial y}A(x,y,u) = A_y(x,y,u) + A_u(x,y,u)\frac{\partial u}{\partial y} = A_y + BA_u,\\ (u_y)_x &= B_x + AB_u,\end{aligned}$$

so setting $(u_x)_y = (u_y)_x$ reveals a "hidden equation", the compatibility condition

$$A_y + BA_u = B_x + AB_u. \tag{1.9}$$

We will prove in §1.9 that the commuting of second-order partials in this case implies that all higher-order mixed partials commute as well, so that there are no further hidden equations. In other words, if (1.9) is an identity in x, y, u, then solving the ODE's (1.7) and (1.8) in succession gives a solution to (1.6), and solutions depend on one constant.

Exercise 1.2.4: Show that, if (1.9) is an identity, then one gets the same solution by first solving for $\tilde{u}(y) = u(0,y)$.

If (1.9) is not an identity, there are several possibilities. If u appears in (1.9), then it gives an equation which every solution to (1.6) must satisfy. Given a point $p = (0,0,u_0)$ at which (1.9) is not an identity, and such that the implicit function theorem may be applied to (1.9) to determine $u(x,y)$ near $(0,0)$, then only this solved-for u can be the solution passing through p. However, it still may not satisfy (1.6), in which case there is no solution through p.

If u does not appear in (1.9), then it gives a relation between x and y, and there is no solution defined on an open set around $(0,0)$.

Remark 1.2.5. For more complicated systems of PDE, it is not as easy to determine if all mixed partials commute. The Cartan-Kähler Theorem (see Chapters 5 and 7) will provide an algorithm which tells us when to stop checking compatibilities.

Exercises 1.2.6:

1. Consider this special case of Example 1.2.3:

$$\begin{aligned}u_x &= A(x,y),\\ u_y &= B(x,y),\end{aligned}$$

where A and B satisfy $A(0,0) = B(0,0) = 0$. Verify that solving the initial value problems (1.7)–(1.8) gives

$$u(x,y) = u_0 + \int_{s=0}^{x} A(s,0)ds + \int_{t=0}^{y} B(x,t)dt. \tag{1.10}$$

Under what condition does this function u satisfy (1.6)? Verify that the resulting condition is equivalent to (1.9) in this special case.

2. Rewrite (1.10) as a line integral involving the 1-form

$$\omega := A(x,y)dx + B(x,y)dy,$$

and determine the condition which ensures that the integral is independent of path.

3. Determine the space of solutions to (1.6) in the following special cases:

(a) $A = -\frac{x}{u}$, $B = -\frac{y}{u}$.
(b) $A = B = \frac{x}{u}$.
(c) $A = -\frac{x}{u}$, $B = y$.

1.3. Introduction to differential equations without coordinates

Example 1.2.3 revisited. Instead of working on $\mathbb{R}^2 \times \mathbb{R}$ with coordinates $(x,y) \times (u)$, we will work on the larger space $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates $(x,y) \times (u) \times (p,q)$, which we will denote $J^1(\mathbb{R}^2, \mathbb{R})$, or J^1 for short. This space, called the *space of* 1*-jets of mappings from* $\mathbb{R}^2$ *to* $\mathbb{R}$, is given additional structure and generalized in §1.9.

Let $u : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. We associate to u the surface in J^1 given by

$$u = u(x,y),\ p = u_x(x,y),\ q = u_y(x,y). \tag{1.11}$$

which we will refer to as the *lift* of u. The graph of u is the projection of the lift (1.11) in J^1 to $\mathbb{R}^2 \times \mathbb{R}$.

We will eventually work on J^1 without reference to coordinates. As a step in that direction, consider the differential forms

$$\theta := du - pdx - qdy, \quad \Omega := dx \wedge dy$$

defined on J^1. Suppose $i : S \hookrightarrow J^1$ is a surface such that $i^*\Omega \neq 0$ at each point of S. Since dx, dy are linearly independent 1-forms on S, we may use x, y as coordinates on S, and the surface may be expressed as

$$u = u(x,y),\ p = p(x,y),\ q = q(x,y).$$

Suppose $i^*\theta = 0$. Then

$$i^*du = pdx + qdy.$$

On the other hand, since u restricted to S is a function of x and y, we have

$$du = u_x dx + u_y dy.$$

Because dx, dy are independent on S, these two equations imply that $p = u_x$ and $q = u_y$ on S. Thus, surfaces $i : S \hookrightarrow J^1$ such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing correspond to lifts of maps $u : U \to \mathbb{R}$.

Now consider the 3-fold $j : \Sigma \hookrightarrow J^1$ defined by the equations

$$p = A(x,y,u), \qquad q = B(x,y,u).$$

Let $i : S \hookrightarrow \Sigma$ be a surface such that $i^*\theta = 0$ and $i^*\Omega$ is nonvanishing. Then the projection of S to $\mathbb{R}^2 \times \mathbb{R}$ is the graph of a solution to (1.6). Moreover, all solutions to (1.6) are the projections of such surfaces, by taking S as the lift of the solution.

Thus we have a correspondence

$$\text{solutions to (1.6)} \Leftrightarrow \text{ surfaces } i : S \hookrightarrow \Sigma \text{ such that } i^*\theta \equiv 0 \text{ and } i^*\Omega \neq 0.$$

On such surfaces, we also have $i^*d\theta \equiv 0$, but

$$\begin{aligned} d\theta &= -dp \wedge dx - dq \wedge dy, \\ j^*d\theta &= -(A_x dx + A_y dy + A_u du) \wedge dx - (B_x dx + B_y dy + B_u du) \wedge dy, \\ i^*d\theta &= (A_y - B_x + A_u B - B_u A) i^*(\Omega). \end{aligned}$$

(To obtain the second line we use the defining equations of Σ and to obtain the third line we use $i^*(du) = Adx + Bdy$.) Because $i^*\Omega \neq 0$, the equation

$$A_y - B_x + A_u B - B_u A = 0 \tag{1.12}$$

must hold on S. This is precisely the same as the condition (1.9) obtained by checking that mixed partials commute.

If (1.12) does not hold identically on Σ, then it gives another equation which must hold for any solution. But since $\dim \Sigma = 3$, in that case (1.12) already describes a surface in Σ. If there is any solution surface S, it must be an open subset of the surface in Σ given by (1.12). This surface will only be a solution if θ pulls back to be zero on it. If (1.12) is an identity, then we may use the Frobenius Theorem (see below) to conclude that through any point of Σ there is a unique solution S (constructed, as in §1.2, by solving a succession of ODE's). In this sense, (1.12) implies that all higher partial derivatives commute.

We have now recovered our observations from §1.2.

The general game plan for treating a system of PDE as an *exterior differential system* (EDS) will be as follows:

One begins with a "universal space" (J^1 in the above example) where the various partial derivatives are represented by independent variables. Then one restricts to the subset Σ of the universal space defined by the system of PDE by considering it as a set of equations among independent variables.

Solutions to the PDE correspond to submanifolds of Σ on which the variables representing what we want to be partial derivatives actually are partial derivatives. These submanifolds are characterized by the vanishing of certain differential forms.

These remarks will be explained in detail in §1.9.

Picard's Theorem revisited. On $\mathbb{R}^2$ with coordinates (x, u), consider $\theta = du - f(x, u)dx$. Then there is a one-to-one correspondence between solutions of the ODE (1.5) and curves $c : \mathbb{R} \to \mathbb{R}^2$ such that $c^*(\theta) = 0$ and $c^*(dx)$ is nonvanishing.

More generally, the flowbox coordinate theorem 1.2.2 implies:

Theorem 1.3.1. *Let M be a C^∞ manifold of dimension m, and let $\theta^1, \dots, \theta^{m-1} \in \Omega^1(M)$ be pointwise linearly independent in some open neighborhood $U \subset M$. Then through $p \in U$ there exists a curve $c : \mathbb{R} \to U$, unique up to reparametrization, such that $c^*(\theta^j) = 0$ for $1 \leq j \leq m - 1$.*

(For a proof, see [**142**].)

The Frobenius Theorem. In §1.9 we will prove the following result, which is a generalization, both of Theorem 1.3.1 and of the asserted existence of solutions to Example 1.2.3 when (1.9) holds, to an existence theorem for certain systems of PDE:

Theorem 1.3.2 (Frobenius Theorem, first version)**.** *Let Σ be a C^∞ manifold of dimension m, and let $\theta^1, \dots, \theta^{m-n} \in \Omega^1(\Sigma)$ be pointwise linearly independent. If there exist 1-forms $\alpha^i_j \in \Omega^1(\Sigma)$ such that $d\theta^j = \alpha^j_i \wedge \theta^i$ for all j, then through each $p \in \Sigma$ there exists a unique n-dimensional manifold $i : N \hookrightarrow \Sigma$ such that $i^*(\theta^j) = 0$ for $1 \leq j \leq m - n$.*

In order to motivate our study of exterior differential systems, we reword the Frobenius Theorem more geometrically as follows: Let Σ be an m-dimensional manifold such that through each point $x \in \Sigma$ there is an n-dimensional subspace $E_x \subset T_x\Sigma$ which varies smoothly with x (such a structure is called a *distribution*). We consider the problem of finding submanifolds $X \subset \Sigma$ such that $T_xX = E_x$ for all $x \in X$.

Consider ${E_x}^\perp \subset T^*_x\Sigma$. Let θ^a_x, $1 \leq a \leq m - n$, be a basis of ${E_x}^\perp$. We may choose the θ^a_x to vary smoothly to obtain $m - n$ linearly independent forms $\theta^a \in \Omega^1(\Sigma)$. Let $\mathcal{I} = \{\theta^1, \dots, \theta^{m-n}\}_{\mathsf{diff}}$ denote the *differential ideal* they generate in $\Omega^*(\Sigma)$ (see §B.4). The submanifolds X tangent to the distribution E are exactly the n-dimensional submanifolds $i : N \hookrightarrow \Sigma$ such that $i^*(\alpha) = 0$ for all $\alpha \in \mathcal{I}$. Call such a submanifold an *integral manifold* of $\mathcal{I}$.

To find integral manifolds, we already know that if there are any, their tangent space at any point $x \in \Sigma$ is already uniquely determined, namely it is E_x. The question is whether these n-planes can be "fitted together" to obtain an n-dimensional submanifold. This information is contained in the derivatives of the θ^a's, which indicate how the n-planes "move" infinitesimally.

If we are to have $i^*(\theta^a) = 0$, we must also have $d(i^*\theta^a) = i^*(d\theta^a) = 0$. If there is to be an integral manifold through x, or even an n-plane $E_x \subset T_x\Sigma$ on which $\alpha\,|_{E_x} = 0$, $\forall \alpha \in \mathcal{I}$, the equations $i^*(d\theta^a) = 0$ cannot impose any additional conditions, i.e., we must have $d\theta^a|_{E_x} = 0$ because we already have a unique n-plane at each point $x \in \Sigma$. To recap, for all a we must have

$$d\theta^a = \alpha^a_1 \wedge \theta^1 + \ldots + \alpha^a_{m-n} \wedge \theta^{m-n} \tag{1.13}$$

for some $\alpha^a_b \in \Omega^1(\Sigma)$, because the forms θ^a_x span $E_x{}^\perp$.

Notation 1.3.3. Suppose $\mathcal{I}$ is an ideal and ϕ and ψ are k-forms. Then we write $\phi \equiv \psi \operatorname{mod} \mathcal{I}$ if $\phi = \psi + \beta$ for some $\beta \in \mathcal{I}$.

Let $\{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{alg}} \subset \Omega^*(\Sigma)$ denote the algebraic ideal generated by $\theta^1, \ldots, \theta^{m-n}$ (see §B.4). Now (1.13) may be restated as

$$d\theta^a \equiv 0 \mod \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{alg}}. \tag{1.14}$$

The Frobenius Theorem states that this necessary condition is also sufficient:

Theorem 1.3.4 (Frobenius Theorem, second version)**.** *Let $\mathcal{I}$ be a differential ideal generated by the linearly independent 1-forms $\theta^1, \ldots, \theta^{m-n}$ on an m-fold Σ, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{diff}}$. Suppose $\mathcal{I}$ is also generated algebraically by $\theta^1, \ldots, \theta^{m-n}$, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{alg}}$. Then through any $p \in \Sigma$ there exists an n-dimensional integral manifold of $\mathcal{I}$. In fact, in a sufficiently small neighborhood of p there exists a coordinate system $y^1, \ldots, y^m$ such that $\mathcal{I}$ is generated by $dy^1, \ldots, dy^{m-n}$.*

We postpone the proof until §1.9.

Definition 1.3.5. We will say a subbundle $I \subset T^*\Sigma$ is *Frobenius* if the ideal generated algebraically by sections of I is also a differential ideal. We will say a distribution $\Delta \subset \Gamma(T\Sigma)$ is *Frobenius* if $\Delta^\perp \subset T^*\Sigma$ is Frobenius. Equivalently (see Exercise 1.3.6.2 below), Δ is Frobenius if $\forall X, Y \in \Delta$, $[X, Y] \in \Delta$, where $[X, Y]$ is the Lie bracket.

If $\{\theta^a\}$ fails to be Frobenius, not all hope is lost for an n-dimensional integral manifold, but we must restrict ourselves to the subset $j : \Sigma' \hookrightarrow \Sigma$ on which (1.14) holds, and see if there are n-dimensional integral manifolds of the ideal generated by $j^*(\theta^a)$ on Σ'. (This was what we did in the special case of Example 1.2.3.)

Exercises 1.3.6:
1. Which of the following ideals are Frobenius?

$$\mathcal{I}_1 = \{dx^1, x^2dx^3 + dx^4\}_{\text{diff}}$$
$$\mathcal{I}_2 = \{dx^1, x^1dx^3 + dx^4\}_{\text{diff}}$$

2. Show that the differential forms and vector field conditions for being Frobenius are equivalent, i.e., $\Delta \subset \Gamma(T\Sigma)$ satisfies $[\Delta, \Delta] \subseteq \Delta$ if and only if $\Delta^\perp \subset T^*\Sigma$ satisfies $d\theta \equiv 0 \mod \Delta^\perp$ for all $\theta \in \Gamma(\Delta^\perp)$.
3. On $\mathbb{R}^3$ let $\theta = Adx + Bdy + Cdz$, where $A = A(x, y, z)$, etc. Assume the differential ideal generated by θ is Frobenius, and explain how to find a function $f(x, y, z)$ such that the differential systems $\{\theta\}_{\text{diff}}$ and $\{df\}_{\text{diff}}$ are equivalent.

1.4. Introduction to geometry without coordinates: curves in $\mathbb{E}^2$

We will return to our study of surfaces in $\mathbb{E}^3$ in Chapter 2. To see how to use moving frames to obtain invariants, we begin with a simpler problem.

Let $\mathbb{E}^2$ denote the oriented Euclidean plane. Given two parametrized curves $c_1, c_2 : \mathbb{R} \to \mathbb{E}^2$, we ask two questions: When does there exist a Euclidean motion $A : \mathbb{E}^2 \to \mathbb{E}^2$ (i.e., a rotation and translation) such that $A(c_1(\mathbb{R})) = c_2(\mathbb{R})$? And, when do there exist a Euclidean motion $A : \mathbb{E}^2 \to \mathbb{E}^2$ and a constant c such that $A(c_1(t)) = c_2(t + c)$ for all t?

Figure 2. Are these two curves equivalent?

Instead of using coordinates at a point, we will use an *adapted frame* , i.e., for each t we take a basis of $T_{c(t)}\mathbb{E}^2$ that is "adapted" to Euclidean geometry. This geometry is induced by the group of Euclidean motions—the changes of coordinates of $\mathbb{E}^2$ preserving the inner product and orientation—which we will denote by $ASO(2)$.

In more detail, the group $ASO(2)$ consists of transformations of the form

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} t^1 \\ t^2 \end{pmatrix} + R \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \tag{1.15}$$

where $R \in SO(2)$ is a rotation matrix. It can be represented as a matrix Lie group by writing

$$ASO(2) = \left\{ M \in GL(3,\mathbb{R}) \,\middle|\, M = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix}, \ \mathbf{t} \in \mathbb{R}^2, R \in SO(2) \right\}. \tag{1.16}$$

Then its action on $\mathbb{E}^2$ is given by $\mathbf{x} \mapsto M\mathbf{x}$, where we represent points in $\mathbb{E}^2$ by $\mathbf{x} = {}^t(1 \quad x^1 \quad x^2)$.

We may define a mapping from $ASO(2)$ to $\mathbb{E}^2$ by

$$\begin{pmatrix} 1 & 0 \\ x & R \end{pmatrix} \mapsto x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.17}$$

which takes each group element to the image of the origin under the transformation (1.15). The fiber of this map over every point is a left coset of $SO(2) \subset ASO(2)$, so $\mathbb{E}^2$, as a manifold, is the quotient $ASO(2)/SO(2)$. Furthermore, $ASO(2)$ may be identified with the bundle of oriented orthonormal bases of $\mathbb{E}^2$ by identifying the columns of the rotation matrix $R = (e_1, e_2)$ with an oriented orthonormal basis of $T_x\mathbb{E}^2$, where x is the basepoint given by (1.17). (Here we use the fact that for a vector space V, we may identify V with T_xV for any $x \in V$.)

Returning to the curve $c(t)$, we choose an oriented orthonormal basis of $T_{c(t)}\mathbb{E}^2$ as follows: A natural element of $T_{c(t)}\mathbb{E}^2$ is $c'(t)$, but this may not be of unit length. So, we take $e_1(t) = c'(t)/|c'(t)|$, and this choice also determines $e_2(t)$. Of course, to do this we must assume that the curve is regular:

Definition 1.4.1. A curve $c(t)$ is said to be *regular* if $c'(t)$ never vanishes. More generally, a map $f : M \to N$ between differentiable manifolds is regular if df is everywhere defined and of rank equal to $\dim M$.

What have we done? We have constructed a map to the Lie group $ASO(2)$ as follows:

$$C : \mathbb{R} \to ASO(2),$$
$$t \mapsto \begin{pmatrix} 1 & 0 \\ c(t) & (e_1(t), e_2(t)) \end{pmatrix}.$$

We will obtain differential invariants of our curve by differentiating this mapping, and taking combinations of the derivatives that are invariant under Euclidean changes of coordinates.

Consider $v(t) = |c'(t)|$, called the *speed* of the curve. It is invariant under Euclidean motions and thus is a differential invariant. However, it is only an invariant of the mapping, not of the image curve (see Exercise 1.4.2.2). The speed measures how much (internal) distance is being distorted under the mapping c.

Consider $\frac{de_1}{dt}$. We must have $\frac{de_1}{dt} = \lambda(t)e_2(t)$ for some function $\lambda(t)$ because $|e_1(t)| \equiv 1$ (see Exercise 1.4.2.1 below). Thus $\lambda(t)$ is a differential invariant, but it again depends on the parametrization of the curve. To determine an invariant of the image alone, we let $\tilde{c}(t)$ be another parametrization of the same curve. We calculate that $\tilde{\lambda}(t) = \frac{\tilde{v}(t)}{v(t)}\lambda(t)$, so we set $\kappa(t) = \frac{\lambda(t)}{v(t)}$. This $\kappa(t)$, called the *curvature* of the curve, measures how much c is infinitesimally moving away from its tangent line at $c(t)$.

A necessary condition for two curves $c, \tilde{c}$ to have equivalent images is that there exists a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $\kappa(t) = \tilde{\kappa}(\psi(t))$. It will follow from Corollary 1.6.13 that the images of curves are locally classified up to congruence by their curvature functions, and that parametrized curves are locally classified by κ, v.

Exercises 1.4.2:

1. Let V be a vector space with a nondegenerate inner product $\langle , \rangle$. Let $v(t)$ be a curve in V such that $F(t) := \langle v(t), v(t)\rangle$ is constant. Show that $v'(t) \perp v(t)$ for all t. Show the converse is also true.
2. Suppose that c is regular. Let $s(t) = \int_0^t |c'(\tau)|d\tau$ and consider c parametrized by s instead of t. Since s gives the length of the image of $c : [0, s] \to \mathbb{E}^2$, s is called an *arclength parameter*. Show that in this preferred parametrization, $\kappa(s) = |\frac{de_1}{ds}|$.
3. Show that $\kappa(t)$ is constant iff the curve is an open subset of a line (if $\kappa = 0$) or circle of radius $\frac{1}{\kappa}$.
4. Let $c(t) = (x(t), y(t))$ be given in coordinates. Calculate $\kappa(t)$ in terms of $x(t), y(t)$ and their derivatives.
5. Calculate the function $\kappa(t)$ for an ellipse. Characterize the points on the ellipse where the maximum and minimum values of $\kappa(t)$ occur.
6. Can $\kappa(t)$ be unbounded if $c(t)$ is the graph of a polynomial?

Exercise 1.4.3 (*Osculating circles*):

(a) Calculate the equation of a circle passing through three points in the plane.

(b) Calculate the equation of a circle passing through two points in the plane and having a given tangent line at one of the points.

Parts (a) and (b) may be skipped; the exercise proper starts here:

(c) Show that for any curve $c \subset \mathbb{E}^2$, at each point $x \in c$ one can define an *osculating circle* by taking the limit of the circle through the three points

$c(t), c(t_1), c(t_2)$ as $t_1, t_2 \to t$. (A line is defined to be a circle of infinite radius.)
(d) Show that one gets the same circle if one takes the limit as $t \to t_1$ of the circle through $c(t)$, $c(t_1)$ that has tangent line at $c(t)$ parallel to $c'(t)$.
(e) Show that the radius of the osculating circle is $1/\kappa(t)$.
(f) Show that if $\kappa(t)$ is monotone, then the osculating circles are nested. ◎

1.5. Submanifolds of homogeneous spaces

Using the machinery we develop in this section and §1.6, we will answer the questions about curves in $\mathbb{E}^2$ posed at the beginning of §1.4. The quotient $\mathbb{E}^2 = ASO(2)/SO(2)$ is an example of a *homogeneous space*, and our answers will follow from a general study of classifying maps into homogeneous spaces.

Definition 1.5.1. Let G be a Lie group, H a closed Lie subgroup, and G/H the set of left cosets of H. Then G/H is naturally a differentiable manifold with the induced differentiable structure coming from the quotient map (see [**77**], Theorem II.3.2). The space G/H is called a *homogeneous space.*

Definition 1.5.2 (Left and right actions)**.** Let G be a group that acts on a set X by $x \mapsto \sigma(g)(x)$. Then σ is called a *left action* if $\sigma(a) \circ \sigma(b) = \sigma(ab)$, or a *right action* if $\sigma(a) \circ \sigma(b) = \sigma(ba)$,

For example, the action of G on itself by left-multiplication is a left action, while left-multiplication by g^{-1} is a right action.

A homogeneous space G/H has a natural (left) G-action on it; the subgroup stabilizing $[e]$ is H, and the stabilizer of any point is conjugate to H. Conversely, a manifold X is a homogeneous space if it admits a smooth transitive action by a Lie group G. If H is the isotropy group of a point $x_0 \in X$, then $X \simeq G/H$, and x_0 corresponds to $[e] \in G/H$, the coset of the identity element. (See [**77, 142**] for additional facts about homogeneous spaces.)

In the spirit of Klein's *Erlanger Programm* (see [**76, 92**] for historical accounts), we will consider G as the group of *motions* of G/H. We will study the geometry of submanifolds $M \subset G/H$, where two submanifolds $M, M' \subset G/H$ will be considered *equivalent* if there exists a $g \in G$ such that $g(M) = M'$.

To determine necessary conditions for equivalence we will find differential invariants as we did in §1.1 and §1.4. (Note that we need to specify whether we are interested in invariants of a mapping or just of the image.) After finding invariants, we will then interpret them as we did in the exercises in §1.4.

We will derive invariants for maps $f : M \to G/H$ by constructing *lifts* $F : M \to G$ as we did for curves in $\mathbb{E}^2$.

Definition 1.5.3. A *lift* of a mapping $f : M \to G/H$ is defined to be a map $F : M \to G$ such that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[d] \\ M \arrow[ur, "F"] \arrow[r, "f"'] & G/H \end{tikzcd}$$

Given a lift F of f, any other lift $\tilde{F} : M \to G$ must be of the form

$$\tilde{F}(x) = F(x)a(x) \tag{1.18}$$

for some map $a : M \to H$.

By associating the value of the lift $F(x)$ with its action on G/H, we may think of choosing a lift to G as analogous to putting a point p in a normalized position, as we did in §1.1.

Given $f : M \to G/H$, we will choose lifts adapted to the infinitesimal geometry. To explain what this statement means, we first remark that in the situations we will be dealing with, the fiber at $x \in G/H$ of the fibration $\pi : G \to G/H$ admits the interpretation of being a subset of the space of *framings* or bases of T_xG/H. Since H fixes the point $[e] \in G/H$, its infinitesimal action on tangent vectors gives a representation $\rho : H \to GL(T_{[e]}G/H)$, called the *isotropy representation* **[153]**. Now fix a reference basis $(v_1, \ldots, v_n)$ of $T_{[e]}G/H$. We identify the H-orbit of this basis with $\pi^{-1}([e])$. Similarly, at other points $x \in G/H$ we have a group conjugate to H acting on T_xG/H.

Thus, a choice of lift may be considered as a choice of framing of G/H along M, and we will make choices that reflect the geometry of M. For example, if $\dim M = n$, we may require the first n basis vectors of T_xG/H to be tangent to M. In the above example of curves in $\mathbb{E}^2$, we also normalized the length of the first basis vector e_1 to be constant.

Once a unique lift is determined, differentiating that lift will provide differential invariants. This is because we can classify maps into G, up to equivalence under left multiplication, using the *Maurer-Cartan form.*

1.6. The Maurer-Cartan form

If you need to brush up on matrix Lie groups and Lie algebras, this would be a good time to consult §A.2 and §A.4.

Definition 1.6.1. Let $G \subseteq GL(n, \mathbb{R})$ be a matrix Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, and let $g = (g^i_j)$ be the matrix-valued function which embeds

G into the vector space $M_{n\times n}$ of $n \times n$ matrices with real entries, with differential $dg_a : T_aG \to T_{g(a)}M_{n\times n} \simeq M_{n\times n}$. We define the *Maurer-Cartan form* of G as

$$\omega_a = g(a)^{-1}dg_a.$$

This is often written

$$\omega = g^{-1}dg.$$

The Maurer-Cartan form is $M_{n\times n}$-valued. In fact, in Exercise 1.6.5 you will show that it takes values in $\mathfrak{g} \subset M_{n\times n}$, i.e., $\omega_a(v) \in \mathfrak{g}$ for all $v \in T_aG$.

Example 1.6.2. Consider $G = SO(2) \subset GL(2,\mathbb{R})$. We may parametrize $SO(2)$ by

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

Then

$$\omega = g^{-1}dg = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}.$$

Definition 1.6.3. A differential form $\alpha \in \Omega^k(G)$ is *left-invariant* if for all $a \in G$, we have $L_a^*(\alpha_g) = \alpha_{a^{-1}g}$, where $L_a : G \to G$ is the diffeomorphism $g \mapsto ag$. (We similarly define left-invariant vector fields and k-vector fields.)

Note that a left-invariant form α is uniquely determined by α_g for any $g \in G$. In this way, the set of left-invariant k-forms may be identified with $\Lambda^k T_g^* G$.

Given an arbitrary Lie group G, we will let $\mathfrak{g}$ denote its Lie algebra, which may be identified with T_eG or with the space of left-invariant vector fields. We generalize the definition of the Maurer-Cartan form to this situation as follows:

Definition 1.6.4. The *Maurer-Cartan form* ω of G is the unique left-invariant $\mathfrak{g}$-valued 1-form on G such that $\omega|_e : T_eG \to \mathfrak{g}$ is the identity map.

Exercise 1.6.5: Show that the two definitions agree when G is a matrix Lie group. In particular, the Maurer-Cartan form of a matrix Lie group is $\mathfrak{g}$-valued.

Remark 1.6.6. When $F : M \to G$ is a lift of a map $f : M \to G/H$, and G is a matrix Lie group, the change in the pullback of the Maurer-Cartan form resulting from a change of lift (1.18) is

$$\tilde{F}^*(\omega) = a^{-1}F^*(\omega)a + a^{-1}da. \tag{1.19}$$

For an abstract Lie group, the analogous formula is

$$\tilde{F}^*(\omega) = \mathrm{Ad}_{a^{-1}}(F^*\omega) + a^*\omega.$$

Definition 1.6.7. Let $\omega = (\omega^i_k)$ and $\eta = (\eta^i_k)$ be matrices whose entries are elements of a vector space V, so that $\omega, \eta \in V \otimes M_{n\times n}$. Define their *matrix wedge product* $\omega \wedge \eta \in \Lambda^2 V \otimes M_{n\times n}$ by

$$(\omega \wedge \eta)^i_j := \omega^i_k \wedge \eta^k_j.$$

More generally, for $\omega \in \Lambda^k V \otimes M_{n\times n}, \eta \in \Lambda^j V \otimes M_{n\times n}$ the same formula yields $\omega \wedge \eta \in \Lambda^{k+j} V \otimes M_{n\times n}$.

One thing that makes the Maurer-Cartan form ω especially useful to work with is that its exterior derivative may be computed algebraically as follows: If G is a matrix Lie group, then

$$d\omega = d(g^{-1}) \wedge dg.$$

To compute $d(g^{-1})$, consider the identity matrix $e = (\delta^i_j)$ as a constant map $G \to M_{n\times n}$ and note that it is the product of two nonconstant functions:

$$0 = d(e) = d(g^{-1}g) = d(g^{-1})g + g^{-1}dg.$$

So, $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ and

$$d\omega = -g^{-1}(dg)g^{-1} \wedge dg = -(g^{-1}dg) \wedge (g^{-1}dg) = -\omega \wedge \omega.$$

Summary 1.6.8. On a matrix Lie group G, the Maurer-Cartan form ω defined by $\omega = g^{-1}dg$ is a left-invariant $\mathfrak{g}$-valued 1-form and satisfies the *Maurer-Cartan equation*:

$$d\omega = -\omega \wedge \omega. \tag{1.20}$$

Definition 1.6.9. If ω, θ are two $\mathfrak{g}$-valued 1-forms, define the $\mathfrak{g}$-valued 2-form $[\omega, \theta]$ by

$$[\omega, \theta](X, Y) = [\omega(X), \theta(Y)] + [\omega(Y), \theta(X)].$$

The Maurer-Cartan equation holds on an abstract Lie group G in the following form:

$$d\omega = -\frac{1}{2}[\omega, \omega]. \tag{1.21}$$

As mentioned above, the Maurer-Cartan form will be our key to classifying maps into homogeneous spaces of G. We first show how it classifies maps into G:

Theorem 1.6.10 (Cartan). *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathfrak{g}$-valued 1-form ϕ satisfying $d\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a neighborhood U of x and a map $f : U \to G$ such that $f^*\omega = \phi$. Moreover, any two such maps f_1, f_2 must satisfy $f_1 = L_a \circ f_2$ for some fixed $a \in G$.*

Corollary 1.6.11. *Given maps $f_1, f_2 : M \to G$, then $f_1^*\omega = f_2^*\omega$ if and only if $f_1 = L_a \circ f_2$ for some fixed $a \in G$.*

Proof of Corollary. Let $\phi = f_1^*\omega$. □

Proof of Theorem 1.6.10. This is a good opportunity to use the Frobenius Theorem.

On $\Sigma = M^n \times G$, let π, ρ denote the projections to each factor and let $\theta = \pi^*(\phi) - \rho^*(\omega)$. Write $\theta = (\theta^i_j)$, and let $I \subset T^*\Sigma$ be the subbundle spanned by the forms θ^i_j. Submanifolds of dimension n to which these forms pull back to be zero are graphs of maps $f : M \to G$ such that $\phi = f^*\omega$. We check the conditions given in the Frobenius Theorem. Calculating derivatives (and omitting the pullback notation), we have

$$\begin{aligned} d\theta &= -\phi \wedge \phi + \omega \wedge \omega \\ &= -\phi \wedge \phi + (\theta - \phi) \wedge (\theta - \phi) \\ &\equiv 0 \operatorname{mod} I. \end{aligned}$$

Thus, the system is Frobenius and there is a unique n-dimensional integral manifold through any $(x, g) \in \Sigma$.

Suppose f_1, f_2 are two different solutions. Say $f_1(x) = g$. Let $a = gf_2(x)^{-1}$. Then the graph of $f = L_a \circ f_2$ passes through (x, g) and $f^*\omega = \phi$. By uniqueness, it follows that $f_1 = L_a \circ f_2$. □

Remark 1.6.12. If we assume in Theorem 1.6.10 that M is connected and simply-connected, then the desired map f may be extended to all of M [**153**].

We may apply Theorem 1.6.10 to classify curves in $\mathbb{E}^2$. In this case, the pullback of the Maurer-Cartan form of $ASO(2) \subset GL(3, \mathbb{R})$ under the lift constructed in §1.4 takes the simple form

$$F^*\omega = \begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & -\kappa\, dt \\ 0 & \kappa\, dt & 0 \end{pmatrix}, \tag{1.22}$$

where t is an arclength parameter.

Corollary 1.6.13. *For curves $c, \tilde{c} \subset \mathbb{E}^2$, if $\kappa(t) = \tilde{\kappa}(t+a)$ for some constant a, then $c, \tilde{c}$ are congruent.*

Exercises 1.6.14:

1. Let $CO(2)$ be the matrix Lie group parametrized by

$$g(\theta, t) = \begin{pmatrix} t\cos\theta & -t\sin\theta \\ t\sin\theta & t\cos\theta \end{pmatrix}, \qquad t \in (0, \infty).$$

Explicitly compute the Maurer-Cartan form and verify the Maurer-Cartan equation for $CO(2)$.

2. Verify (1.19).
3. Verify (1.22) and complete the proof of Corollary 1.6.13.
4. Show that (1.21) coincides with (1.20) when G is a matrix Lie group.
5. Let $\mathfrak{g}$ be a vector space with basis X_B, $1 \leq B \leq \dim \mathfrak{g}$, and a multiplication given by $[X_A, X_B] = c^C_{AB} X_C$ on the basis and extended linearly. Determine necessary and sufficient conditions on the constants c^A_{BC} implying that, with this bracket, $\mathfrak{g}$ is a Lie algebra.
6. On a Lie group G with Maurer-Cartan form ω, show that

$$d\omega_e(X, Y) = [X, Y].$$

Conclude that $d\omega = 0$ iff G is abelian.

7. On a Lie group G with Lie algebra $\mathfrak{g}$, let $\{e_A\}$ be a basis of $\mathfrak{g}$ and write $\omega = \omega^A e_A$. (Note that each 1-form ω^A is left-invariant.) Write $d\omega^A = -\tilde{C}^A_{BC}\omega^B \wedge \omega^C$, where $\tilde{C}^A_{BC} = -\tilde{C}^A_{CB}$. Show that the coefficients $\tilde{C}^A_{BC}$ are constants, and determine the set of equations that these constants must satisfy because $d^2 = 0$. Relate these equations to your answer to problem 5.

1.7. Plane curves in other geometries

Equivalence of holomorphic mappings under fractional linear transformations. Here is an example of a study of curves in a less familiar homogeneous space, the complex projective line $\mathbb{CP}^1$. To find differential invariants in such situations, we generally seek a uniquely defined lift that renders the pullback of the Maurer-Cartan form as simple as possible. Then, after finding differential invariants, we interpret them.

Definition 1.7.1. A *fractional linear transformation* (FLT) is a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ given in terms of homogeneous coordinates ${}^t[x, y]$ by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right], \qquad \text{with } ad - bc = 1.$$

The group of FLT's is $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm \mathrm{Id}\}$, which acts transitively on $\mathbb{CP}^1$. Thus, $\mathbb{CP}^1$ is a homogeneous space $PSL(2,\mathbb{C})/P$, where

$$P = \left[\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right] \in PSL(2,\mathbb{C})$$

is the stabilizer of ${}^t[1, 0]$. Although $PSL(2,\mathbb{C})$, as presented, is not a matrix Lie group, we may avoid problems by localizing as follows:

If $\Delta \subset \mathbb{C} \subset \mathbb{CP}^1$ is a domain, then $PSL(2,\mathbb{C})$ acts on maps $f : \Delta \to \mathbb{C}$ by

$$f \mapsto \frac{af + b}{cf + d}.$$

(Since we will be working locally, there is no harm in considering f as a map to $\mathbb{C}$; then to think of f as a map to $\mathbb{CP}^1$, write it as ${}^t[f, 1]$.) Suppose

$f, g : \Delta \to \mathbb{C}$ are two holomorphic maps with nonzero first derivatives. When are these locally equivalent via a fractional linear transformation, i.e., when does $g = A \circ f$ for some FLT A? (One can ask the same question in the real category for analytic maps $f, g : (0,1) \to \mathbb{RP}^1$, and the answer will be the same.)

Note that in this example the target is of the same dimension as the source of the mapping, so we cannot expect an analogue of curvature, but there will be an analogue of speed.

The coordinate approach to getting invariants would be to use an FLT to normalize the map at some point z_0, say by requiring $f(z_0) = 0$, $f'(z_0) = 1$ and $f''(z_0) = 0$. Since this is exactly the extent of normalization that $PSL(2,\mathbb{C})$ can achieve, then $f'''(z_0)$ must be an invariant. Of course, this is valid only at the point z_0.

Instead we construct a lift to $PSL(2,\mathbb{C})$, which we will treat as $SL(2,\mathbb{C})$ in order to be working with a matrix Lie group. As a first try, let

$$F = \begin{pmatrix} f & -1 \\ 1 & 0 \end{pmatrix}.$$

where the projection to $\mathbb{CP}^1$ is the equivalence class of the first column. Any other lift $\tilde{F}$ of f must be of the form $\tilde{F}(z) = F(z)A(z)$, where

$$A(z) = \begin{pmatrix} a(z) & b(z) \\ 0 & a^{-1}(z) \end{pmatrix}, \qquad a(z) \neq 0.$$

We want to pick functions a, b to obtain a new lift whose Maurer-Cartan form is as simple as possible. We have

$$\begin{aligned}
\tilde{F}^{-1}d\tilde{F} &= A^{-1}F^{-1}dFA + A^{-1}dA \\
&= \left\{ \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -f' & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & -a'a^{-2} \end{pmatrix} \right\} dz \\
&= \begin{pmatrix} abf' + a^{-1}a' & a^{-2}ba' + a^{-1}b' + b^2 f' \\ -a^2 f' & -(abf' + a^{-1}a') \end{pmatrix} dz.
\end{aligned}$$

Choose $a(z) = \pm 1/\sqrt{|f'(z)|}$ with the same sign as $f'(z)$. Then $\tilde{F}^{-1}d\tilde{F}$ takes the form

$$\begin{pmatrix} * & * \\ \mp 1 & * \end{pmatrix} dz.$$

(This is the analogue of setting $f'(z_0) = 1$ in the coordinate approach.) The function b is still free. We use it to set the diagonal term in the pullback of the Maurer-Cartan form to zero, i.e., to set $abf' + a^{-1}a' = 0$. This implies

$$b = -\frac{a'}{a^2 f'} = \pm \frac{f''}{2|f'|^{3/2}}.$$

Now our lift is unique and of the form

$$\begin{pmatrix} 0 & \frac{1}{2}\mathcal{S}_f(z) \\ \mp 1 & 0 \end{pmatrix} dz,$$

where

$$\mathcal{S}_f(z) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is a differential invariant, called the *Schwarzian derivative* [**1**]. The ambiguity of the $\pm$ is due to the fact that $G = PSL(2,\mathbb{C})$, not $SL(2,\mathbb{C})$.

Exercises 1.7.2:

1. Show that if f is an FLT then $S_f \equiv 0$. So, just as the curvature of a curve in $\mathbb{R}^2$ measures the failure of a curve to be a line, $\mathcal{S}_f(z)$ is an infinitesimal measure of the failure of a holomorphic map to be an FLT. Since FLT's map circles to circles, $\mathcal{S}_f$ may be thought of as measuring how much circles are being distorted under f.

2. Calculate S_f for $f = ae^{bz}$, and $f = x^n$. How to these compare asymptotically? What does this say about how circles are distorted as one goes out to infinity?

Exercises on curves in other plane geometries.

Exercises 1.7.3:

1. *(Curves in the special affine plane)* We consider the geometry of curves that are equivalent up to translations and area-preserving linear transformations of $\mathbb{R}^2$. These transformations are given by the matrix group

$$ASL(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} \middle| \, x \in \mathbb{R}^2, A \in SL(2,\mathbb{R}) \right\},$$

acting on $\mathbb{R}^2$ in the same way as $ASO(2)$ acts in §1.4. Since the origin is fixed by the subgroup $SL(2,\mathbb{R})$, in this context we will relabel $\mathbb{R}^2$ as the *special affine plane* $S\mathbb{A}^2 = ASL(2,\mathbb{R})/SL(2,\mathbb{R})$.

(a) Find differential invariants for curves in $S\mathbb{A}^2$. (As with the Euclidean case, one can consider invariants of a parametrized curve or invariants of just the image curve.)

(b) What are the image curves with invariants zero? The image curves with constant invariants?

(c) Let κ_A denote the differential invariant that distinguishes image curves. Interpret $\kappa_A(t)$ in terms of an osculating curve, as we did with the osculating circles to a curve in §1.4.

(d) The preferred frame will lead to a unique choice of e_2. Give a geometric interpretation of e_2. ◎

2. *(Curves in the projective plane)* Carry out the analogous exercise for curves in the projective plane $\mathbb{P}^2 = GL(3)/P$, where P is the subgroup preserving a line. Show that the curves with zero invariants are the projective

lines and plane conics. Derive the *Monge equation* $((y'')^{\frac{-2}{3}})''' = 0$, characterizing plane conics, by working in a local adapted coordinate system. Note that one may do this exercise over $\mathbb{R}$ or $\mathbb{C}$.
3. Carry out the analogous exercise for curves in the conformal plane $ACO(2)/CO(2)$, where equivalence is up to translations, rotations and dilations.
4. Carry out the analogous exercise for curves in $\mathbb{L}^2 = ASO(1,1)/SO(1,1)$, where $SO(1,1)$ is the subgroup of $GL(2,\mathbb{R})$ preserving the quadratic form $Q = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that there will be three distinct types of curves: *spacelike* curves, where $Q(c'(t), c'(t)) > 0$; *timelike* curves, where $Q(c'(t), c'(t)) < 0$; and *lightlike* curves, where $Q(c'(t), c'(t)) = 0$. What are the curves with constant invariants?

1.8. Curves in $\mathbb{E}^3$

The group $ASO(3)$ and its Maurer-Cartan form. The group $ASO(3)$ is the set of transformations of $\mathbb{E}^3$ of the form $\mathbf{x} \mapsto \mathbf{t} + R\mathbf{x}$, i.e.,

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix} + R \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

where $R \in SO(3)$ is a rotation matrix. Like $ASO(2)$, it may be represented as a matrix Lie group by writing

$$ASO(3) = \left\{ M \in GL(4,\mathbb{R}) \,\middle|\, M = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix},\ \mathbf{t} \in \mathbb{R}^3, R \in SO(3) \right\}. \tag{1.23}$$

The action on $\mathbb{E}^3$ is given by $\mathbf{x} \mapsto M\mathbf{x}$, where we represent points in $\mathbb{E}^3$ by $\mathbf{x} = {}^t\begin{pmatrix} 1 & x^1 & x^2 & x^3 \end{pmatrix}$.

Having expressed $ASO(3)$ as in (1.23), we may express an arbitrary element of its Lie algebra $\mathfrak{aso}(3)$ as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x^1 & 0 & -x^2_1 & -x^3_1 \\ x^2 & x^2_1 & 0 & -x^3_2 \\ x^3 & x^3_1 & x^3_2 & 0 \end{pmatrix}, \qquad x^i, x^i_j \in \mathbb{R}.$$

In this presentation, the Maurer-Cartan form of $ASO(3)$ is

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}, \tag{1.24}$$

where $\omega^i, \omega^i_j \in \Omega^1(ASO(3))$. Recall from §1.6 that the forms ω^i, ω^i_j are left-invariant, and are a basis for the space of left-invariant 1-forms on $ASO(3)$.

We identify $ASO(3)$ with the space of *oriented orthonormal frames* of $\mathbb{E}^3$, as follows. Denote $g \in ASO(3)$ by a 4-tuple of vectors (warning: this is not a presentation as a matrix Lie group),

$$g = (x, e_1, e_2, e_3), \tag{1.25}$$

where $x \in \mathbb{E}^3$ corresponds to translation by x, and $\{e_1, e_2, e_3\}$ is an oriented orthonormal basis of $T_x\mathbb{E}^3$ which corresponds to the rotation $R = (e_1, e_2, e_3) \in SO(3)$.

With this identification, we obtain geometric interpretations of the left-invariant forms. Substituting (1.25) and (1.24) into $dg = g\,\omega$, and considering the first column, gives

$$dx = e_i\omega^i. \tag{1.26}$$

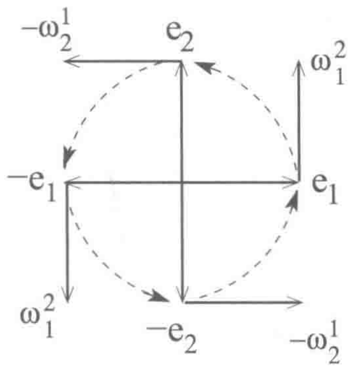
Thus, ω^i has the geometric interpretation of measuring the infinitesimal motion of the point x in the direction of e_i. More precisely, if $x(t) \in \mathbb{E}^3$ lifts to

$$C(t) = (x(t), e_1(t), e_2(t), e_3(t)) \in ASO(3),$$

then $\omega^i(C'(t)) = \langle x'(t), e_i \rangle$. Similarly, ω^i_j measures the infinitesimal motion of e_j toward e_i, because the other columns of $dg = g\,\omega$ show that

$$de_j = e_i\omega^i_j. \tag{1.27}$$

That these motions are infinitesimal rotations is reflected in the relation $\omega^i_j = -\omega^j_i$, as illustrated by the following picture:



Recall from Appendix B that a form $\alpha \in \Omega^1(P)$ on a bundle $\pi : P \to M$ is *semi-basic* for π if $\alpha(v) = 0$ for all $v \in \ker \pi_*$.

Proposition 1.8.1. *The forms ω^i, $1 \leq i \leq 3$, are semi-basic for the projection $ASO(3) \to \mathbb{E}^3$.*

Proof. Let $C(t) = (x(t), e_1(t), e_2(t), e_3(t)) \subset ASO(3)$ be a curve in a fiber. We need to show that $\omega^i(C'(t)) = 0$. If $C(t)$ stays in one fiber, then $\frac{dx}{dt} = 0$,

but equation (1.27) shows $\frac{dx}{dt} = C'(t) \lrcorner\, dx = \omega^j(C'(t))e_j(t)$. The result follows because the e_j are linearly independent. $\square$

Differential invariants of curves in $\mathbb{E}^3$. We find differential invariants of a regular curve $c : \mathbb{R} \to \mathbb{E}^3$. For simplicity, we only consider the image curve, so we can and will assume $|c'(t)| \equiv 1$. Consequently, we have $c'' \perp c'$ (see Exercise 1.4.2.1). To obtain a lift $C : \mathbb{R} \to ASO(3)$ we may take $e_1(t) = c'(t)$, $e_2(t) = c''(t)/|c''(t)|$ and this determines $e_3(t)$. Our adaptations have the effect that $C^*(\omega^1)$ is nonvanishing and $C^*(\omega^2) = C^*(\omega^3) = 0$. In terms of the Maurer-Cartan form, we have:

$$d(x(t), e_1(t), e_2(t), e_3(t))$$

$$= (x(t), e_1(t), e_2(t), e_3(t))\; C^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega_1^2 & -\omega_1^3 \\ 0 & \omega_1^2 & 0 & -\omega_2^3 \\ 0 & \omega_1^3 & \omega_2^3 & 0 \end{pmatrix}. \tag{1.28}$$

Exercise 1.8.2: Show that $C^*(\omega_1^3) = 0$.

All forms pulled back to $\mathbb{R}$ will be multiples of ω^1, as $\omega^1 = dt$ furnishes a basis of $T^*\mathbb{R}^1$ at each point. (We continue our standard abuse of notation, writing ω^1 instead of $C^*(\omega^1)$.) So, we may write $\omega_1^2 = \kappa(t)\omega^1$ and $\omega_2^3 = \tau(t)\omega^1$, where $\kappa(t), \tau(t)$ are functions called the *curvature* and *torsion* of the curve. Traditionally one writes $e_1 = T$, $e_2 = N$, $e_3 = B$; then (1.28) yields the *Frenet equations*

$$d(T, N, B) = (T, N, B) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} dt.$$

Curves with $\kappa \equiv 0$ are lines, and we may think of κ as a measurement of the failure of the curve to be a line. Curves with $\tau \equiv 0$ lie in a plane, and we may think of τ as measuring the failure of a curve to lie in a plane. In contrast to the example of plane curves, we needed a third-order invariant (the torsion) to determine a unique lift in this case.

Theorem 1.3.1 implies that one can specify any functions $(\kappa(t), \tau(t))$, and there will be a curve having these as curvature and torsion (because on $ASO(3) \times \mathbb{R}$ the forms $\omega^1 - d(t), \omega^3, \omega^2, \omega_1^2 - \kappa(t)\omega^1, \omega_1^3, \omega_2^3 + \tau(t)\omega^1$ satisfy the hypotheses of the theorem). If the functions are nowhere vanishing the curve will be unique up to congruence (see the exercises below).

Remark 1.8.3. Defining N as the unit vector in the direction of $c''(t)$ means that κ cannot be negative, and N (along with the binormal B and the torsion) is technically undefined at inflection points along the curve (i.e., points where $c''(t) = 0$). However, it is still possible to smoothly extend the

frame (T, N, B) across inflection points, while satisfying the Frenet equations for smooth functions $(\kappa(t), \tau(t))$ where κ is allowed to change sign (see the discussion in [**12**]). Such frames are sometimes called *generalized Frenet frames*, and it is in this sense that ODE existence theorems provide a framed curve with given curvature and torsion functions.

Exercises 1.8.4:

1. Using Corollary 1.6.11, show that if $c, \tilde{c}$ are curves with $\kappa(t) = \tilde{\kappa}(t)$, $\tau(t) = \tilde{\tau}(t)$, then $c, \tilde{c}$ differ by a rigid motion.

2. Show that a curve $c \subset \mathbb{R}^3$ has constant κ and τ if and only if there exists a line $l \subset \mathbb{E}^3$ with the property that every normal line of c intersects l orthogonally. (A normal line is the line through $c(t)$ in the direction of $N(t)$.)

3. *(Bertrand curves)* In the previous exercise we characterized curves with constant invariants. Here we study the next simplest case, when there is a linear relation among the curvature and torsion, i.e., constants a, b, c such that $a\kappa(t) + b\tau(t) = c$ for all t.

(a) Show that if such a linear relation holds, then there exists a second curve $\overline{c}(t)$ with the same normal line as $c(t)$ for all t.

(b) Show moreover that the distance between the points $c(t)$ and $\overline{c}(t)$ is constant. ⊚

(c) Characterize the curves c where there exists more than one curve $\overline{c}$ with this property.

4. Derive invariants for curves in $\mathbb{E}^n$. How many derivatives does one need to take to obtain a complete set of invariants?

5. *(Curves on spheres)* Show that a curve c with $\kappa, \tau \neq 0$ is contained in a sphere if and only if $\rho^2 + \sigma^2$ is constant, where $\rho = 1/\kappa$ and $\sigma = \rho'/\tau$. ⊚

6. Let $\mathbb{L}^3 = ASO(2,1)/SO(2,1)$, where $SO(2,1)$ is the subgroup of $GL(3, \mathbb{R})$ preserving

$$Q = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Find differential invariants of curves in $\mathbb{L}^3$. (As before, curves may be spacelike, timelike, or lightlike.) What are the curves with constant invariants?

1.9. Exterior differential systems and jet spaces

In §1.3, we saw how a system of ODE or PDE could be replaced by an ideal of differential forms, and solutions became submanifolds on which the forms pulled back to be zero. We now formalize this perspective, defining exterior differential systems with and without independence condition.

Exterior differential systems with independence condition.

Definition 1.9.1. An *exterior differential system with independence condition* on a manifold Σ consists of a differential ideal $\mathcal{I} \subset \Omega^*(\Sigma)$ and a differential n-form $\Omega \in \Omega^n(\Sigma)$ defined up to scale. This Ω, or its equivalence class $[\Omega]$ up to scale, is called the *independence condition.*

(See §B.4 for a discussion of differential ideals.)

Definition 1.9.2. An *integral manifold* (or solution) of the system $(\mathcal{I}, \Omega)$ is an immersed n-fold $f : M^n \to \Sigma$ such that $f^*(\alpha) = 0\ \forall \alpha \in \mathcal{I}$ and $f^*(\Omega) \neq 0$ at each point of M.

We also define the notion of an infinitesimal solution:

Definition 1.9.3. Let V be a vector space, and let $G(n, V)$ denote the Grassmannian of n-planes through the origin in V (see Appendix A). We say $E \in G(n, T_x\Sigma)$ is an *integral element* of $(\mathcal{I}, \Omega)$ if $\Omega|_E \neq 0$ and $\alpha|_E = 0\ \forall \alpha \in \mathcal{I}$. We let $\mathcal{V}_n(\mathcal{I}, \Omega)_x$ denote the space of integral elements of $(\mathcal{I}, \Omega)$ at $x \in \Sigma$.

Integral elements are the potential tangent spaces to integral manifolds, in the sense that the integral manifolds of an exterior differential system are the immersed submanifolds $M \subset \Sigma$ such that T_xM is an integral element for all $x \in M$.

Exercise 1.9.4: Let $\mathcal{I}^n = \mathcal{I} \cap \Omega^n(\Sigma)$. Show that $\mathcal{V}_n(\mathcal{I}, \Omega)_x = \{E \in G_n(T_x\Sigma) \mid \alpha|_E = 0\ \ \forall \alpha \in \mathcal{I}^n\}$.

We now explain how to rephrase any system of PDE as an exterior differential system with independence condition using the language of jet bundles.

Jets.

Definition 1.9.5. Let t be a coordinate on $\mathbb{R}$ and let $k \geq 0$. Two differentiable maps $f, g : \mathbb{R} \to \mathbb{R}$ with $f(0) = g(0) = 0$ are said to have the same *k-jet* at 0 if

$$\frac{df}{dt}(0) = \frac{dg}{dt}(0),\ \frac{d^2f}{dt^2}(0) = \frac{d^2g}{dt^2}(0), \ldots, \frac{d^kf}{dt^k}(0) = \frac{d^kg}{dt^k}(0).$$

Let M, N are differentiable manifolds and $f, g : M \to N$ be two maps. Then f and g are said to have the same k-jet at $p \in M$ if

i. $f(p) = g(p) = q$, and

ii. for all maps $u : \mathbb{R} \to M$ and $v : N \to \mathbb{R}$ with $u(0) = p$, the differentiable maps $v \circ f \circ u$ and $v \circ g \circ u$ have the same k-jet at 0.

Exercise 1.9.6: Show that to determine if $f, g : M \to N$ have the same k-jet at p, it is sufficient to check derivatives up to order k with respect to coordinate directions in any pair of local coordinate systems around p and q.

Having the same k-jet at p is an equivalence relation on smooth maps. We denote the equivalence class of f by $j^k_p(f)$, the space k-jets where p maps to q by $J^k_{pq}(M, N)$ and the space of all k-jets of all maps from M to N by $J^k(M, N)$. This is a smooth manifold, with local coordinates as follows:

Suppose M has local coordinates x^i and N local coordinates u^a. Then $J^k(M, N)$ has coordinates x^i, u^a, p^a_i, p^a_{ij}, ..., $p^a_{i_1,\dots,i_k}$. We will abbreviate this as (x^i, u^a, p^a_I), where I is a multi-index of length up to k whose entries range between 1 and $\dim N$. Then the point $j^k_{x_0}(f) \in J^k(M, N)$ has coordinates x^i_0, $u^a = f^a(x_0)$ and $p^a_I = \frac{\partial^{|I|} f^a}{\partial x^I}(x_0)$ for $1 \le |I| \le k$.

Furthermore, because we assume f is smooth, we may take the entries in I to be nondecreasing. For example, coordinates on $J^2(\mathbb{R}^2, \mathbb{R})$ would be $x^1, x^2, u^1, p^1_1, p^1_2, p^1_{11}, p^1_{12}$ and p^1_{22}.

Exercise 1.9.7: Calculate the dimensions of (a) $J^2(\mathbb{R}^3, \mathbb{R})$, (b) $J^3(\mathbb{R}^2, \mathbb{R}^2)$, (c) $J^k(\mathbb{R}^n, \mathbb{R}^n)$.

Note that $T^*_x M = J^1_{x,0}(M, \mathbb{R})$, $T_x M = J^1_{0,x}(\mathbb{R}, M)$, and, in general, $J^k(M, N)$ is a bundle over M (as well as over $M \times N$). Any map $f : M \to N$ induces a section $p \mapsto j^k_p(f)$ of this bundle, called the *lift of the graph* of f.

Canonical contact systems. On $J^k(M, N)$ there is a canonical EDS with independence condition, called the *contact system*, whose integral manifolds are the lifts of graphs of maps $f : M \to N$ to $J^k(M, N)$. We now describe this system in the local coordinates used above.

Let $\Omega := dx^1 \wedge \ldots \wedge dx^n$ and let $\mathcal{I}$ be the ideal generated differentially by the 1-forms

$$
\begin{aligned}
\theta^a &:= du^a - p^a_i dx^i, \\
\theta^a_i &:= dp^a_i - p^a_{ij} dx^j, \\
&\ \ \vdots \\
\theta^a_{i_1,\dots,i_{k-1}} &:= dp^a_{i_1,\dots,i_{k-1}} - p^a_{i_1,\dots,i_k} dx^{i_k},
\end{aligned}
\tag{1.29}
$$

which we will call *contact forms*. (Note the summation on i_k.) We will use multi-index notation to abbreviate the forms in (1.29) as

$$\theta^a_I := dp^a_I - p^a_{Ij} dx^j.$$

The system $(\mathcal{I},\Omega)$ on $J^k(M.N)$ is defined globally and is independent of the coordinates chosen. It is called the canonical *contact system* on $J^k(M,N)$. Its integral manifolds are exactly the lifts of graphs $\Gamma_f = \{(x,f(x)) \mid x \in M\} \subset M \times N$ of mappings $f : M \to N$ to $J^k(M,N)$. To see this, let $i : X \hookrightarrow J^k$ be an n-dimensional integral manifold with local coordinates $x^1,\dots,x^n$. On X, $u = u(x^1,\dots,x^n)$, $p^a_i = p^a_i(x^1,\dots,x^n)$, etc., and $i^*(\theta^a) = 0$ implies that $p^a_i = \frac{\partial u^a}{\partial x^i}$ for all $1 \le i \le n$. Similarly, the vanishing of the other forms in the ideal force the other jet coordinates to be the higher derivatives of u.

How to express any PDE system as an EDS with independence condition. Given a kth-order system of PDE for maps $f : \mathbb{R}^n \to \mathbb{R}^s$,

$$F^r\left(x^i, u^a, \frac{\partial^{|I|}u^a}{\partial x^I}\right) = 0, \qquad 1 \le r \le R, \quad 1 \le |I| \le k, \tag{1.30}$$

we define a submanifold $\Sigma \subset J^k$ by the equations $F^r(x^i,u^a,p^a_I) = 0$. The lifts of solutions of (1.30) are precisely the integral manifolds of the pullback of the contact system to Σ. Note that Ω tells us what the independent variables should be.

Standard abuse of notation. Given an inclusion $i : M \hookrightarrow \Sigma$, instead of writing $i^*(\theta) = 0$ or $i^*(\Omega) \neq 0$ we will often simply say respectively "$\theta = 0$ on M"or "$\Omega \neq 0$ on M".

Exterior differential systems. We generalize our notion of exterior differential systems with independence condition as follows:

Definition 1.9.8. An *exterior differential system* on a manifold Σ is a differential ideal $\mathcal{I} \subset \Omega^*(\Sigma)$. An *integral manifold* of the system $\mathcal{I}$ is an immersed submanifold $f : M \to \Sigma$ such that $f^*(\alpha) = 0 \ \forall \alpha \in \mathcal{I}$.

Note that for an exterior differential system, not only do we do not specify the analog of independent variables, but we do not even specify a required dimension for integral manifolds.

We define a k-dimensional *integral element* of $\mathcal{I}$ at $x \in \Sigma$ to be an $E \in G(k, T_x\Sigma)$ such that $\alpha|_E = 0 \ \forall \alpha \in \mathcal{I}$. Let $\mathcal{V}_k(\mathcal{I})_x$ denote the space of k-dimensional integral elements to $\mathcal{I}$ at x.

Exercise 1.9.9: Let $\mathcal{I} = \{x^1dx^2, dx^3\}_{\mathsf{diff}}$ be an exterior differential system on $\mathbb{R}^3$. Calculate $\mathcal{V}_1(\mathcal{I})_{(1,1,1)}$, $\mathcal{V}_1(\mathcal{I})_{(0,0,0)}$, $\mathcal{V}_2(\mathcal{I})_{(1,1,1)}$ and $\mathcal{V}_2(\mathcal{I})_{(0,0,0)}$.

Let $\Omega = adx^1 + bdx^2 + edx^3$, where a,b,c are constants, not all zero. Calculate $\mathcal{V}_1(\mathcal{I},\Omega)_{(1,1,1)}$ and $\mathcal{V}_1(\mathcal{I},\Omega)_{(0,0,0)}$.

Proof of the Frobenius Theorem. We now prove Theorem 1.3.4, which we restate as follows:

Theorem 1.9.10 (Frobenius Theorem, second version). *Let $\mathcal{I}$ be a differential ideal generated by the linearly independent 1-forms $\theta^1, \ldots, \theta^{m-n}$ on an m-fold Σ, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{diff}}$. Suppose $\mathcal{I}$ is also generated algebraically by $\theta^1, \ldots, \theta^{m-n}$, i.e., $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{alg}}$. Then through any $p \in \Sigma$ there exists an n-dimensional integral manifold of $\mathcal{I}$. In fact, in a sufficiently small neighborhood of p there exists a coordinate system $y^1, \ldots, y^m$ such that $\mathcal{I}$ is generated by $dy^1, \ldots, dy^{m-n}$.*

Proof. We follow the proof in [**20**], as that proof will get us used to calculations with differential forms.

We proceed by induction on n. If $n = 1$, then the distribution defines a line field and we are done by Theorem 1.3.1. Assume that the theorem is true up to $n - 1$, and we will show that it is true for n.

Let $x : M \to \mathbb{R}$ be a smooth function such that $\theta^1 \wedge \ldots \wedge \theta^{m-n} \wedge dx \neq 0$ on a neighborhood U of p, and consider the ideal $\mathcal{I}' = \{\theta^1, \ldots, \theta^{m-n}, dx\}_{\mathsf{diff}}$. Since $\mathcal{I} = \{\theta^1, \ldots, \theta^{m-n}\}_{\mathsf{diff}}$ is Frobenius, $\mathcal{I}'$ is also Frobenius. By our induction hypothesis, there exist local coordinates $(y^1, \ldots, y^m)$ such that $\mathcal{I}' = \{dy^1, \ldots, dy^{m-n+1}\}_{\mathsf{alg}}$.

At this point we have an $(n-1)$-dimensional integral manifold of $\mathcal{I}'$ (hence, also of $\mathcal{I}$) passing through p, obtained by setting $y^1, \ldots, y^{m-n+1}$ equal to the appropriate constants. We want to enlarge it to an n-dimensional integral manifold.

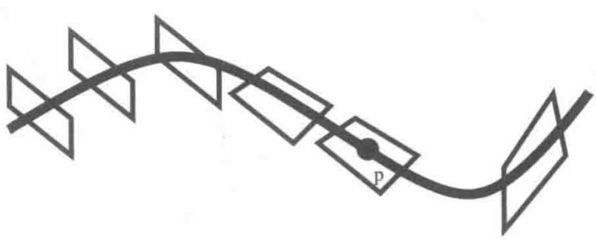


Figure 3. Curve with plane field to be enlarged to surface

Let $1 \leq i, j \leq n - m$. We may write

$$dx = a_i dy^i + a_{m-n+1} dy^{m-n+1},$$
$$\theta^i = c^i_j dy^j + c^i_{m-n+1} dy^{m-n+1},$$

where $a_i, a_{m-n+1}, c^i_j, c^i_{m-n+1}$ are smooth functions. Without loss of generality, we may assume $\partial x / \partial y^{m-n+1} \neq 0$, so we may rewrite the second line

as

$$\theta^i = \tilde{c}^i_j dy^j + f^i dx$$

for some smooth functions $\tilde{c}^i_j, f^i$. The matrix of functions $\tilde{c}^i_j$ is invertible at each point in a (possibly smaller) neighborhood $\tilde{U}$ of p, so locally we may take a new set of generators for $\mathcal{I}$, of the form

$$\tilde{\theta}^i = dy^i + e^i dx$$

for some smooth functions e^i, and with $\theta^i = \tilde{c}^i_j \tilde{\theta}^j$. Then $d\tilde{\theta}^i = de^i \wedge dx$ and, since $\mathcal{I}$ is Frobenius,

$$de^i \wedge dx \equiv 0 \operatorname{mod}\{\tilde{\theta}^i\}.$$

Hence

$$de^i = a dx + b^i_j dy^j$$

for some functions a, b^i_j. In particular, the e^i are functions of the y^j and x only, and it follows that the $\tilde{\theta}^i$ are defined in terms of the variables $y^1, \ldots, y^{m-n+1}$ only.

Let $V \subset \tilde{U}$ be the submanifold through p obtained by setting y^{m-n+2} through y^m constant. Then $\mathcal{I}|_V$ is a codimension-one Frobenius system. Hence there are coordinates $(\tilde{y}^1, \ldots, \tilde{y}^{m-n+1})$ on V that are functions of the $y^1, \ldots, y^{m-n+1}$, such that $\mathcal{I}|_V$ is generated by $d\tilde{y}^1, \ldots, d\tilde{y}^{m-n}$. These relationships extend to $\tilde{U}$, so that

$$(\tilde{y}^1, \ldots, \tilde{y}^{m-n+1}, y^{m-n+2}, \ldots, y^m)$$

is the desired coordinate system. □

Symplectic manifolds, contact manifolds and their EDS's. What follows are two examples of classical exterior differential systems and a complete local description of their integral manifolds.

Symplectic manifolds. Let $\Sigma = \mathbb{R}^{2n}$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ and let

$$\phi = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{1.31}$$

Consider the exterior differential system $\mathcal{I} = \{\phi\}_{\mathsf{diff}}$.

Exercises 1.9.11:

1. Show that at any point $\frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n}$ is an integral element of $\mathcal{I}$.
2. Show that any graph $y^j = f^j(x^1, \ldots, x^n)$ of the form $y^j = f^j(x^j)$ for each j is an integral manifold.

We claim there are no $(n+1)$-dimensional integral elements for $\mathcal{I}$. First, it is easy to check that ϕ is *nondegenerate*, i.e., $\phi(v, w) = 0$ for all vectors $w \in T_p\Sigma$ only if $v = 0$. Now suppose $E = \{v_1, \ldots, v_{n+1}\}$ were an integral element at some point $p \in \Sigma$. Then the nondegeneracy of ϕ implies that

the forms $\alpha_j = v_j \lrcorner \phi \in T_p^*\Sigma$ are linearly independent. However, this is impossible since $\alpha_i|_E = 0$ for every α_i.

Exercise 1.9.12: Alternatively, show that there are no $(n+1)$-dimensional integral elements to $\mathcal{I}$ by relating ϕ to the standard inner product $\langle , \rangle$ on $\mathbb{R}^{2n}$. Namely, let $\phi(v,w) = \langle v, Jw \rangle$, where J is the standard complex structure defined in Exercise A.3.1. Then, if E is a integral element, show that $\langle , \rangle$ must be degenerate on $E \cap J(E)$.

An even-dimensional manifold with closed nondegenerate 2-form is called a *symplectic manifold*, and the 2-form is called a *symplectic form*. The following theorem shows that the above example on $\mathbb{R}^{2n}$ is quite general.

Notation 1.9.13. For $\omega \in \Omega^2(M)$, we will write ω^r for the r-fold wedge product $\omega \wedge \omega \wedge \cdots \wedge \omega$ of ω with itself.

Theorem 1.9.14 (Darboux). *Suppose a closed 2-form $\omega \in \Omega^2(M^n)$ is such that $\omega^r \neq 0$ but $\omega^{r+1} = 0$ in some neighborhood $U \subset M$. Then there exists a coordinate system $w^1, \ldots, w^n$, possibly in a smaller neighborhood, such that*

$$\omega = dw^1 \wedge dw^2 + \ldots + dw^{2r-1} \wedge dw^{2r}.$$

In particular, ω takes the form (1.31) *when $n = 2r$.*

Darboux's Theorem implies that all symplectic manifolds are locally equivalent, in contrast to Riemannian manifolds (see 2.6.12). Globally this is not at all the case, and the study of the global geometry of symplectic manifolds is an active area of research (see [**47**], for example).

Example 1.9.15. Given any differentiable manifold M, the cotangent bundle T^*M is canonically a symplectic manifold.

Let $\pi : T^*M \to M$ be the projection and let $\alpha \in \Omega^1(T^*M)$ be the *canonical 1-form* defined as follows: $\alpha(v)_{(x,u)} = u(\pi_*(v))$, where $x \in M$ and $u \in T_x^*M$. If M has local coordinates x^i, then T^*M has local coordinates (x^i, y_j) such that if $u \in T_x^*M$ then $u = \sum_j y_j(x,u)dx^j$. So, in these coordinates $\alpha = \sum_j y_j dx^j$. Hence, $\omega = d\alpha$ is a symplectic form on T^*M.

Contact manifolds.

Exercise 1.9.16: Consider the contact system on $J^1(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^{2n+1}$ with coordinates $(z, x^1, \ldots, x^n, y^1, \ldots, y^n)$. Here $\theta = dz - \Sigma_i y^i dx^i$ generates the contact system $\mathcal{I} = \{\theta\}_{\mathsf{diff}}$

(a) Show that at any point $\frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n}$ is an integral element.

(b) Show that any graph $z = h(x^1, \ldots, x^n), y^j = f^j(x^1, \ldots, x^n)$ such that $f^j = \partial h / \partial x^j$ is an integral manifold. (In fact, all n-dimensional integral manifolds are locally of this form.)

(c) Show that there are no $(n+1)$-dimensional integral elements for $\mathcal{I}$.

Again, this example is general:

Theorem 1.9.17 (Pfaff). *Let M be a manifold of dimension $n+1$, let $\theta \in \Omega^1(M)$ and $\mathcal{I} = \{\theta\}_{\mathsf{diff}}$. Let $r \in \mathbb{N}$ be such that $(d\theta)^r \wedge \theta \neq 0$ but $(d\theta)^{r+1} \wedge \theta = 0$ in some neighborhood $U \subset M$. Then there exists a coordinate system $w^0, \ldots, w^n$, possibly in a smaller neighborhood, such that $\mathcal{I}$ is locally generated by*

$$\tilde{\theta} = dw^0 + w^{r+1}dw^1 + \ldots + w^{2r}dw^r$$

(i.e., θ is a nonzero multiple of $\tilde{\theta}$ on U). In fact, there exist coordinates $y^0, \ldots, y^n$ such that

$$\theta = \begin{cases} y^0dy^1 + y^2dy^3 + \ldots + y^{2r}dy^{2r+1} & \text{if } (d\theta)^{r+1} \neq 0, \\ dy^1 + y^2dy^3 + \ldots + y^{2r}dy^{2r-1} & \text{if } (d\theta)^{r+1} = 0, \end{cases}$$

on U.

If $n = 2r+1$, then the Pfaff Theorem implies that M is locally diffeomorphic to the jet bundle $J^1(\mathbb{R}^r, \mathbb{R})$ and $\tilde{\theta}$ is the pullback of the standard contact form. Thus, r-dimensional integral manifolds of $\mathcal{I}$ are given in the coordinates $w^0, \ldots, w^n$ by

$$\begin{aligned} w^0 &= f(w^1, \ldots, w^r), \\ w^{r+1} &= \frac{\partial f}{\partial w^1}, \\ &\vdots \\ w^{2r} &= \frac{\partial f}{\partial w^r}. \end{aligned}$$

A 1-form θ on a $(2n+1)$-dimensional manifold Σ is called a *contact form* if it is as nondegenerate as possible, i.e., if $\theta \wedge (d\theta)^n$ is nonvanishing.

Since we use the one form θ to define an EDS, we really only care about it up to multiplication by a nonvanishing function. A *contact manifold* is defined to be a manifold with a contact form, defined up to scale. This generalizes the contact structure on $J^1(M, \mathbb{R})$, our first example of a contact manifold.

Example 1.9.18. The projectivized tangent bundle $\mathbb{P}TM$ may be given the structure of a contact manifold, by taking the distribution $\alpha^\perp \subset T(TM)$ and projecting to $\mathbb{P}TM$.

Exercise 1.9.19 (Normal form for degenerate contact forms)**:** On $\mathbb{R}^3$, consider a 1-form θ such that $\theta \wedge d\theta = f\Omega$, where Ω is a volume form and f is a function such that $df|_p \neq 0$ whenever $f(p) = 0$. Show that there are coordinates (x, y, z), in a neighborhood of any such point, such that $\theta = dz - y^2dx$.

Chapter 2

Euclidean Geometry and Riemannian Geometry

In this chapter we return to the study of surfaces in Euclidean space $\mathbb{E}^3 = ASO(3)/SO(3)$. Our goal is not just to understand Euclidean geometry, but to develop techniques for solving equivalence problems for submanifolds of arbitrary homogeneous spaces. We begin with the problem of determining if two surfaces in $\mathbb{E}^3$ are locally equivalent up to a Euclidean motion. More precisely, given two immersions $f, \tilde{f} : U \to \mathbb{E}^3$, where U is a domain in $\mathbb{R}^2$, when do there exist a local diffeomorphism $\phi : U \to U$ and a fixed $A \in ASO(3)$ such that $\tilde{f} \circ \phi = A \circ f$? Motivated by our results on curves in Chapter 1, we first try to find a complete set of *Euclidean differential invariants* for surfaces in $\mathbb{E}^3$, i.e., functions $I_1, \ldots, I_r$ that are defined in terms of the derivatives of the parametrization of a surface, with the property that $f(U)$ differs from $\tilde{f}(U)$ by a Euclidean motion if and only if $(\tilde{f} \circ \phi)^* I_j = f^* I_j$ for $1 \leq j \leq r$.

In §2.1 we derive the Euclidean differential invariants Gauss curvature K and mean curvature H using moving frames. Unlike with curves in $\mathbb{E}^3$, for surfaces in $\mathbb{E}^3$ there is not always a unique lift to $ASO(3)$, and we are led to define the *space of adapted frames*. (Our discussion of adapted frames for surfaces in $\mathbb{E}^3$ is later generalized to higher dimensions and codimensions in §2.5.) We calculate the functions H, K for two classical classes of surfaces in §2.2; developable surfaces and surfaces of revolution, and discuss basic properties of these surfaces.

Scalar-valued differential invariants turn out to be insufficient (or at least not convenient) for studying equivalence of surfaces and higher-dimensional submanifolds, and we are led to introduce *vector bundle valued invariants*. This study is motivated in §2.4 and carried out in §2.5, resulting in the definitions of the first and second fundamental forms, I and II. In §2.5 we also interpret II and Gauss curvature, define the Gauss map and derive the Gauss equation for surfaces.

Relations between intrinsic and extrinsic geometry of submanifolds of Euclidean space are taken up in §2.6, where we prove Gauss's theorema egregium, derive the Codazzi equation, discuss frames for C^∞ manifolds and Riemannian manifolds, and prove the fundamental lemma of Riemannian geometry. We include many exercises about connections, curvature, the Laplacian, isothermal coordinates and the like. We conclude the section with the fundamental theorem for hypersurfaces.

In §2.7 and §2.8 we discuss two topics we will need later on, space forms and curves on surfaces. In §2.9 we discuss and prove the Gauss-Bonnet and Poincaré-Hopf theorems. We conclude this chapter with a discussion of non-orthonormal frames in §2.10, which enables us to finally prove the formula (1.3) and show that surfaces with H identically zero are minimal surfaces.

The geometry of surfaces in $\mathbb{E}^3$ is studied further in §3.1 and throughout Chapters 5–7. Riemannian geometry is discussed further in Chapter 8.

2.1. Gauss and mean curvature via frames

Guided by Cartan's Theorem 1.6.11, we begin our search for differential invariants of immersed surfaces $f : U^2 \to \mathbb{E}^3$ by trying to find a lift $F : U \to ASO(3)$ which is adapted to the geometry of $M = f(U)$. The most naïve lift would be to take

$$F(p) = \begin{pmatrix} 1 & 0 \\ f(p) & \mathrm{Id} \end{pmatrix}.$$

Any other lift $\tilde{F}$ is of the form

$$\tilde{F} = F \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

for some map $R : U \to SO(3)$.

Let $x = f(p)$; then $T_x\mathbb{E}^3$ has distinguished subspaces, namely $f_*(T_pU)$ and its orthogonal complement. We use our rotational freedom to adapt to this situation by requiring that e_3 always be normal to the surface, or equivalently that $\{e_1, e_2\}$ span T_xM. This is analogous to our choice of coordinates at our preferred point in Chapter 1, but is more powerful since it works on an open set of points in U.

We will call a lift such that e_3 is normal to T_xM a *first-order adapted lift*, and continue to denote such lifts by F. Our adaptation implies that

$$F^*(\omega^3) = 0, \tag{2.1}$$

$$F^*(\omega^1 \wedge \omega^2) \neq 0 \text{ at each point.} \tag{2.2}$$

The equation $dx = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3$ (see (1.26)) shows that (2.1) can be interpreted as saying that x does not move in the direction of e_3 to first order, and (2.2) implies that to first order x may move independently towards e_1 and e_2.

Let $\pi : \mathcal{F}^1 \to U$ denote the bundle whose fiber over $x \in U$ is the set of oriented orthonormal bases (e_1, e_2, e_3) of $T_{f(x)}\mathbb{E}^3$ such that $e_3 \perp T_{f(x)}M$. The first-order adapted lifts are exactly the sections of $\mathcal{F}^1$. By fixing a reference frame at the origin in $\mathbb{E}^3$, $ASO(3)$ may be identified as the bundle of all oriented orthonormal frames of $\mathbb{E}^3$, and $\mathcal{F}^1$ is a subbundle of $f^*(ASO(3))$. Throughout this chapter we will not distinguish between U and M when the distinction is unimportant. In particular, we will usually consider $\mathcal{F}^1$ as a bundle over M.

Consequences of our adaptation. Thanks to the Maurer-Cartan equation (1.20), we may calculate the derivatives of the left-invariant forms on $ASO(3)$ algebraically:

$$d\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}.$$

Write $i : \mathcal{F}^1 \hookrightarrow ASO(3)$ for the inclusion map. By our definition of $\mathcal{F}^1$, $i^*\omega^3 = 0$, and hence

$$0 = i^*(d\omega^3) = -i^*(\omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2). \tag{2.3}$$

By (2.2), $i^*\omega^1$ and $i^*\omega^2$ are independent, and we can apply the Cartan Lemma A.1.9 to the right hand side of (2.3). We obtain

$$i^*\begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} i^*\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \tag{2.4}$$

where $h_{ij} = h_{ji}$ are some functions defined on $\mathcal{F}^1$. This $h = (h_{ij})$ is analogous to the Hessian at the origin in (1.1), but it has the advantage of being defined on all of $\mathcal{F}^1$.

Given an adapted lift $F : U \to \mathcal{F}^1$, we have

$$F^* \begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = h_F \, F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where $h_F = F^*(h)$. We now determine the invariance of h_F. Since F is uniquely defined up to a rotation in the tangent plane to M, all other possible adapted lifts are of the form

$$\widetilde{F} = F \begin{pmatrix} 1 & & \\ & R & \\ & & 1 \end{pmatrix} = Fr, \tag{2.5}$$

where $R : U \to SO(2)$ is an arbitrary smooth function.

We compare $\widetilde{F}^*(\omega) = \widetilde{F}^{-1} d\widetilde{F}$ with $F^*(\omega) = F^{-1}dF$:

$$\begin{aligned}
\widetilde{F}^{-1} d\widetilde{F} &= (Fr)^{-1} d(Fr) = r^{-1}(F^{-1}dF)r + r^{-1}F^{-1}F dr \\
&= \begin{pmatrix} 1 & & \\ & R^{-1} & \\ & & 1 \end{pmatrix} F^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega_1^2 & -\omega_1^3 \\ \omega^2 & \omega_1^2 & 0 & -\omega_2^3 \\ \omega^3 & \omega_1^3 & \omega_2^3 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & R & \\ & & 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & & \\ & R^{-1}dR & \\ & & 0 \end{pmatrix}.
\end{aligned}$$

In particular,

$$\widetilde{F}^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = R^{-1} F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \qquad \widetilde{F}^*(\omega_1^3, \omega_2^3) = F^*(\omega_1^3, \omega_2^3) R.$$

Since $R^{-1} = {}^tR$, we conclude that

$$h_{\tilde{F}} = R^{-1} h_F R. \tag{2.6}$$

Thus, the properties of h_F that are invariant under conjugation by a rotation matrix are invariants of the mapping f. The functions $\frac{1}{2}\operatorname{trace}(h_F)$ and $\det(h_F)$ generate the ideal of functions on h_F that are invariant under (2.6). They are respectively called the *mean curvature* (first defined by Sophie Germain), denoted by H, and the *Gauss curvature* (first defined by a mathematician with better p.r.), denoted by K. We see immediately that for two surfaces to be congruent it is necessary that they must have the same Gauss and mean curvature functions at corresponding points, thus recovering our observations of §1.1.

Another perspective. Instead of working with lifts to $\mathcal{F}^1$, one could work with $h : \mathcal{F}^1 \to S^2\mathbb{R}^2$ directly, calculating how h varies as one moves in the fiber.

Let k_1, k_2 denote the eigenvalues of h; for the sake of definiteness, say $k_1 \geq k_2$. These are called the *principal curvatures* of $M \subset \mathbb{E}^3$, and are also differential invariants. However, H, K are more natural invariants, because, e.g., the Gauss curvature plays a special role in the intrinsic geometry of the surface; see for example Theorem 2.6.2 below.

Note also that if M is smooth, then H, K are smooth functions on M while k_1 and k_2 may fail to be differentiable at points where $k_1 = k_2$, which are called *umbilic points.*

Exercises 2.1.1:

1. Let $\tilde{F}$ be as in (2.5) and let $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Calculate $\tilde{F}^*\omega_1^2$ in terms of θ and $F^*\omega_1^2$.
2. Show that H, K are invariants of the image of f. ◎
3. Express k_1, k_2 in terms of H and K. ◎

Example 2.1.2 (Surfaces with $H = K = 0$)**.** If $H = K = 0$, then the matrix h is zero and ω_1^3, ω_2^3 vanish. So, on $\Sigma = ASO(3)$ define $\mathcal{I} = \{\omega^3, \omega_1^3, \omega_2^3\}_{\mathsf{diff}}$ with independence condition $\Omega = \omega^1 \wedge \omega^2$.

Suppose $f : U \to \mathbb{E}^3$ gives a surface M with H, K identically zero. Such a surface is (as you may already have guessed) a subset of a plane. For, if $F : U \to ASO(3)$ is a first-order adapted frame for a surface with $H = K = 0$, then $F(U)$ will be an integral surface of $(\mathcal{I}, \Omega)$. Note that $de_3 = -\omega_1^3 e_1 - \omega_2^3 e_2 = 0$, so e_3 is constant for such lifts. Therefore, for all $x \in M$ there is a fixed vector e_3 such that $e_3 \perp T_x M$, and thus M is contained in a plane perpendicular to e_3.

2.2. Calculation of H and K for some examples

The Helicoid. Let $\mathbb{R}^2$ have coordinates (s, t), fix a constant $a > 0$ and consider the mapping $f : \mathbb{R}^2 \to \mathbb{E}^3$ defined by

$$f(s, t) = (s\cos t, s\sin t, at).$$

The image surface is called the *helicoid.*

Exercise 2.2.1: Draw the helicoid. Note that the z-axis is contained in the surface, as is a horizontal line emanating out from each point on the z-axis, and this line rotates as we move up the z-axis.

We compute a first-order adapted frame for the helicoid. Note that

$$\begin{aligned} f_s &= (\cos(t), \sin(t), 0), \\ f_t &= (-s\sin(t), s\cos(t), a). \end{aligned}$$

so $\langle f_s, f_t\rangle = 0$ and we may take

$$
\begin{aligned}
e_1 &= \frac{f_s}{|f_s|} = (\cos(t), \sin(t), 0),\\
e_2 &= \frac{f_t}{|f_t|} = \frac{1}{\sqrt{s^2+a^2}}(-s\sin(t), s\cos(t), a),\\
e_3 &= e_1 \times e_2 = \frac{1}{\sqrt{s^2+a^2}}(a\sin(t), -a\cos(t), s).
\end{aligned}
\tag{2.7}
$$

Since $df = f_s ds + f_t dt = f^*(\omega^1)e_1 + f^*(\omega^2)e_2$, we obtain (omitting the f^* from the notation here and in what follows)

$$
\begin{aligned}
\omega^1 &= ds,\\
\omega^2 &= (s^2+a^2)^{\frac{1}{2}}dt.
\end{aligned}
\tag{2.8}
$$

Next, we calculate

$$
\begin{aligned}
de_3 = &\Big(-s(s^2+a^2)^{-3/2}(a\sin(t), -a\cos(t), s) + ((s^2+a^2)^{-1/2}(0,0,1)\Big)\, ds\\
&+ (s^2+a^2)^{-1/2}(a\cos(t), a\sin(t), 0)dt.
\end{aligned}
$$

So, using (2.7), (2.8), we obtain

$$
\begin{aligned}
\omega^3_1 &= -a(s^2+a^2)^{-1}\omega^2,\\
\omega^3_2 &= -a(s^2+a^2)^{-1}\omega^1,
\end{aligned}
$$

and conclude that $H(s,t) \equiv 0$ and $K(s,t) = -\frac{a^2}{(s^2+a^2)^2}$.

Surfaces with H identically zero are called *minimal surfaces* and are discussed in more detail in §2.10 and §6.4.

Developable surfaces. A surface $M^2 \subset \mathbb{E}^3$ is said to be *developable* if it is describable as (a subset of) the union of tangent rays to a curve. (Developable surfaces are also called *tangential surfaces.*)

Let $c : \mathbb{R} \to \mathbb{E}^3$ be a regular parametrized curve, and consider the surface $f : \mathbb{R}^2 \to \mathbb{E}^3$ defined by $(u,v) \mapsto c(u) + vc'(u)$. Since

$$
df = (c'(u) + vc''(u))du + c'(u)dv,
$$

we see that f is regular, i.e., df is of maximal rank, when $c''(u)$ is linearly independent from $c'(u)$ and $v \neq 0$. (We will assume $v > 0$.) Note that the tangent space, spanned by $c'(u), c''(u)$, is independent of v.

Assume that c is parametrized by arclength. Then $\langle c'(u), c''(u)\rangle = 0$, and we can take

$$
\begin{aligned}
e_1(u,v) &= c'(u),\\
e_2(u,v) &= c''(u)/||c''(u)||.
\end{aligned}
$$

Using $df = \omega^1 e_1 + \omega^2 e_2$, we calculate

$$\omega^1 = du + dv,$$
$$\omega^2 = v\kappa(u)du,$$

where $\kappa(u)$ is the curvature of c.

Note that our frame is the same as if we were to take an adapted framing of c (as in §1.8), so we have

$$de_3 = (-\tau(u)e_2)du.$$

Thus,

$$\omega^3_1 = 0, \qquad \omega^3_2 = \frac{\tau(u)}{\kappa(u)v}\omega^2,$$

showing that $H(u,v) = \frac{\tau(u)}{2\kappa(u)v}$ and $K \equiv 0$.

Surfaces with K identically zero are called *flat*, and we study their geometry more in §2.4.

Developable surfaces are also examples of *ruled surfaces* (as is the helicoid). A surface is *ruled* if through any point of the surfaces there passes a straight line (or line segment) contained in the surface.

Surfaces of revolution. Let $U \subset \mathbb{R}^2$ be an open set with coordinates u, v and let $f : U \to \mathbb{E}^3$ be a map of the form

$$x(u,v) = r(v)\cos(u),$$
$$y(u,v) = r(v)\sin(u),$$
$$z(u,v) = t(v),$$

where r, t are smooth functions. The resulting surface is called a *surface of revolution* because it is constructed by rotating a generating curve (e.g., in the xz-plane) about the z-axis. Call the image M.

Assuming that the generating curve is regular, we can choose v to be an arclength parameter, so that $(r'(v))^2 + (t'(v))^2 = 1$. Let

$$\begin{aligned} e_1 &= (-\sin u, \cos u, 0), \\ e_2 &= (r'(v)\cos u, r'(v)\sin u, t'(v)). \end{aligned} \tag{2.9}$$

Note that $e_j \in \Gamma(U, f^*(T\mathbb{E}^3))$.

Exercises 2.2.2:

1. (a) Show that e_1, e_2 in (2.9) is an orthonormal basis of $T_{f(u,v)}M$.
(b) Calculate e_3 such that e_1, e_2, e_3 is an orthonormal basis of $T_{f(u,v)}\mathbb{E}^3$.
2. Considering this frame as a lift $F : U \to \mathcal{F}^1$, calculate the pullback of the Maurer-Cartan forms in terms of du, dv.
3. Calculate the Gauss and mean curvature functions of M.

4. Consider the surfaces of revolution generated by the following data. In each case, describe the surface geometrically. (Take time out to draw some pictures and have fun!) Calculate H, K and describe their asymptotic behavior.
(a) $r(v) =$ constant, $t(v) = v$.
(b) $r(v) = av$, $t(v) = bv$, where $a^2 + b^2 = 1$.
(c) $r(v) = \cos v$, $t = \sin v$.
(d) The generating curve in the xz plane is a parabola, e.g. $x - bz^2 = c$.
(e) The generating curve is a hyperbola, e.g. $x^2 - bz^2 = c$.
(f) The generating curve is an ellipse, e.g. $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$.
5. Find all surfaces of revolution with $K \equiv 0$. Give a geometric construction of these surfaces. ⊚
6. Find all surfaces of revolution with $K \equiv 1$ that intersect the $x - y$ plane perpendicularly. (Your answer should involve an integral and the choice of one arbitrary constant.) Which of these are complete? ⊚

2.3. Darboux frames and applications

Recall that $k_1 \geq k_2$ are the eigenvalues of the second fundamental form matrix h. Away from umbilic points (points where $k_1 = k_2$), k_1 and k_2 are smooth functions, and we may further adapt frames by putting h in the form

$$h = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

because a real symmetric matrix is always diagonalizable by a rotation matrix. In this case F is uniquely determined. We will call such a framing a *Darboux* or *principal framing.*

Notation. In general, we will express the derivative of a function u on a framed surface as $du = u_1\omega^1 + u_2\omega^2$, where $u_j = e_j(u)$. In particular, write $dk_j = k_{j,1}\omega^1 + k_{j,2}\omega^2$ to represent the derivatives of the k_j in Exercises 2.3.1 below.

Exercises 2.3.1:
1. Let $c \subset \mathbb{E}^2$ be the curve defined by intersecting M with the plane through x parallel to e_1, e_3. Show that the curvature of c at x is k_1.
2. Calculate $F^*(\omega^2_1)$ in a Darboux framing as a function of the principal curvatures and their derivatives.
3. Suppose that X_1, X_2 are vector fields on V such that $f_*X_i = e_i$. Show that

$$[X_1, X_2] = -\frac{k_{1,2}}{k_1 - k_2}X_1 + \frac{k_{2,1}}{k_1 - k_2}X_2. \text{ ⊚}$$

4. Find all surfaces in $\mathbb{E}^3$ with $k_1 \equiv k_2$, i.e., surfaces where each point is an umbilic point.

5. Derive the *Codazzi equation* for Darboux frames, i.e., show that k_1, k_2 satisfy the differential equation

(2.10)
$$-k_1k_2 = \frac{1}{(k_1-k_2)^2}\left((k_1-k_2)(k_{2,11}-k_{1,22}) + k_{1,2}k_{2,2} + k_{2,1}k_{1,1}\right). \quad \circledcirc$$

This to some extent addresses the existence question for principal curvature functions. Namely, two functions $k_1(u,v)$, $k_2(u,v)$ that are never equal cannot be the principal curvature functions of some embedding of $U \to \mathbb{E}^3$ unless they satisfy the Codazzi equation. In particular, surfaces with both H and K constant must be either flat or totally umbilic.

6. Using the Codazzi equation, show that if $k_1 > k_2$ everywhere, and if there exists a point p at which k_1 has a local maximum and k_2 a local minimum, then $K(p) \leq 0$.

Even among surfaces of revolution, there are an infinite number of non-congruent surfaces with $K \equiv 1$. We will see in Example 5.8.2 and again in §6.4 that surfaces with constant $K > 0$ are even more flexible in general. Thus, the following theorem might come as a surprise:

Theorem 2.3.2. *If $M^2 \subset \mathbb{E}^3$ is compact, without boundary, and has constant Gauss curvature $K > 0$, then M is the round sphere.*

Exercise 2.3.3: Prove the theorem. $\circledcirc$

2.4. What do H and K tell us?

Since Darboux frames provide a unique lift for M and well-defined differential invariants, it is natural to pose the question:

> **Question:** Are surfaces $M^2 \subset \mathbb{E}^3$ with no umbilic points locally determined, up to a Euclidean motion, by the functions H and K?

The answer is **NO!** Consider the following example:

The Catenoid. Let $\mathbb{R}^2$ have coordinates (u,v), let $a > 0$ be a constant, and consider the following mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$:

$$f(u,v) = (a\cosh v \cos u, a \cosh v \sin u, av)$$

The image is called the *catenoid*.

Exercise 2.4.1: Draw the catenoid. Calculate the mean and Gauss curvature functions by choosing an adapted orthonormal frame and differentiating as we did in §2.2. $\circledcirc$

Now consider the map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $s = a\sinh v$, $t = u$.

Exercise 2.4.2: Show that $g^*(K_{cat}) = K_{helicoid}$, and of course the mean curvature functions match up as well.

Although we have the same Gauss and mean curvature functions, the helicoid is ruled, and it is not hard to check that the catenoid contains no line segments. Since a Euclidean transformation takes lines to lines, we see it is impossible for the helicoid to be equivalent to the catenoid via a Euclidean motion.

We will see in Example 6.4.24 that, given a non-umbilic surface with constant mean curvature, there are a circle's worth of non-congruent surfaces with the same Gauss curvature function and mean curvature as the given surface. On the other hand, the functions H and K are usually sufficient to determine M up to congruence. Those surfaces for which this is not the case either have constant mean curvature, or belong to a finite-dimensional family called *Bonnet surfaces*, after Ossian Bonnet, who first investigated them; see §6.4 for more discussion.

We will also see, in §2.6, that using slightly more information, namely vector bundle valued differential invariants, one can always determine local equivalence of surfaces from second-order information.

Flat surfaces. Recall that a surface is *flat* if $K \equiv 0$. The name is justified by Theorem 2.6.2, which implies that the intrinsic geometry of such surfaces is the same as that of a plane, and also by the following exercise:

Exercise 2.4.3: Show that if M is flat, there exist local coordinates x^1, x^2 on M and an orthonormal frame (e_1, e_2, e_3) such that $\omega^1 = dx^1, \omega^2 = dx^2$. ⊚

Here are some examples of flat surfaces:

Cylinders. Let $C \subset \mathbb{E}^2 \subset \mathbb{E}^3$ be a plane curve parametrized by $c(u)$, and assume X is a unit normal to $\mathbb{E}^2$. Let $f(u,v) = c(u) + vX$.

Exercise 2.4.4: Find a Darboux framing for the cylinder and calculate its Gauss and mean curvature functions.

Cones. Let $C \subset \mathbb{E}^3$ be a curve parametrized by $c(u)$, and let $p \in \mathbb{E}^3 \backslash C$. Let $f(u,v) = c(u) + v(p - c(u))$. The resulting surface is called the *cone* over c with vertex p.

Exercise 2.4.5: Show that cones are indeed flat. ⊚

Remark 2.4.6. It turns out the property of being flat is invariant under a larger group than $ASO(3)$, and flat surfaces are best studied by exploiting this larger group. This topic will be taken up in Chapter 3, where we classify all flat surfaces: in the projective complex analytic category they are either cones, cylinders, or tangential surfaces to a curve. Even in the C^∞ category,

the only complete flat surfaces are cylinders; see [**142**], where there are also extensive comments about the local characterization of flat C^∞-surfaces.

2.5. Invariants for n-dimensional submanifolds of $\mathbb{E}^{n+s}$

We already saw that for surfaces, the functions H, K alone were not sufficient to determine equivalence. We now begin the study of vector bundle valued functions as differential invariants for submanifolds of (oriented) Euclidean space $\mathbb{E}^{n+s} = ASO(n+s)/SO(n+s)$.

Let $(x, e_1, \ldots, e_{n+s})$ denote an element of $ASO(n+s)$. Define the projection

$$\pi : ASO(n+s) \to \mathbb{E}^{n+s},$$
$$(x, e_1, \ldots, e_{n+s}) \mapsto x.$$

Given an n-dimensional submanifold $M \subset \mathbb{E}^{n+s}$, let $\pi : \mathcal{F}^1 \to M$ denote the subbundle of $ASO(n+s)|_M$ of oriented *first-order adapted frames* for M, whose fiber over a point $x \in M$ is the set of oriented orthonormal bases such that $e_1, \ldots, e_n$ are tangent to M (equivalently, $e_{n+1}, \ldots, e_{n+s}$ are normal to M).

Using index ranges $1 \leq i, j, k \leq n$ and $n+1 \leq a, b \leq s$, we write the Maurer-Cartan form on $ASO(n+s)$ as

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix}.$$

On $\mathcal{F}^1$, continuing our standard abuse in omitting pullbacks from the notation, $\omega^a = 0$ and thus $d\omega^a = -\omega^a_j \wedge \omega^j = 0$, which implies

$$\omega^a_j = h^a_{ij}\omega^j,$$

for some functions $h^a_{ij} = h^a_{ji} : \mathcal{F}^1 \to \mathbb{R}$.

We seek quantities that are invariant under motions in the fiber $(\mathcal{F}^1)_x$. The motions in the fiber of $\mathcal{F}^1$ are given by left-multiplication by

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g^i_j & 0 \\ 0 & 0 & u^a_b \end{pmatrix}, \tag{2.11}$$

where $(g^i_k) \in SO(n)$ and $(u^a_b) \in SO(s)$. If $\tilde{f} = Rf$, then

$$\tilde{h}^a_i j = (u^{-1})^a_b g^k_i g^l_j h^b_{kl}. \tag{2.12}$$

In this situation, if we were to look for scalar functions that are constant on the fibers, we would get a mess, but there are simple vector bundle valued

functions that are constant on the fibers. Recall our general formula (1.19) for how the Maurer-Cartan form changes under a change of lift. Under a motion (2.11) we have

$$\omega^i \mapsto (g^{-1})^i_j \omega^j, \qquad e_a \mapsto u^b_a e_b, \qquad \omega^a_i \mapsto g^i_j (u^{-1})^a_b \omega^b_j. \tag{2.13}$$

Let NM denote the *normal bundle* of M, the bundle with fiber $N_x M = (T_x M)^\perp \subset T_x \mathbb{E}^{n+s}$. Define

$$\begin{aligned}\widetilde{II} &:= \omega^a_j \omega^j \otimes e_a \\ &= h^a_{ij} \omega^i \omega^j \otimes e_a \in \Gamma(\mathcal{F}^1, \pi^*(S^2 T^* M \otimes NM)),\end{aligned}$$

where we write $\omega^i \omega^j$ for the symmetric product $\omega^i \circ \omega^j$. Then (2.13) shows that $\widetilde{II}$ is constant on fibers and thus is basic, i.e., if $s_1, s_2 : M \to \mathcal{F}^1$ are any two sections, then $s_1^*(\widetilde{II}) = s_2^*(\widetilde{II})$.

Proposition/Definition 2.5.1. *$\widetilde{II}$ descends to a well-defined differential invariant*

$$II \in \Gamma(M, S^2 T^* M \otimes NM)$$

called the (Euclidean) second fundamental form of M.

When studying surfaces, we failed to mention the vector bundle valued first-order invariants of submanifolds described in the following exercises.

Exercises 2.5.2:

1. Consider

$$\tilde{I} := \sum_i \omega^i \omega^i \in \Gamma(\mathcal{F}^1, \pi^*(S^2 T^* M)).$$

Verify that $\tilde{I}$ descends to a well-defined differential invariant

$$I \in \Gamma(M, S^2 T^* M),$$

which is called the *first fundamental form* or *Riemannian metric* of M.

2. Show that $\mathrm{dvol} := \omega^1 \wedge \ldots \wedge \omega^n$ is invariant under motions in the fiber and descends to a well-defined invariant, called the *volume form* of M. Show that it is indeed the volume form induced by the Riemannian metric I. (Recall that an inner product on a vector space V induces an inner product on $\Lambda^n V$, and thus a volume form up to a sign.)

Interpretations of II and K. We now give a more geometrical definition of the second fundamental form for surfaces in $\mathbb{E}^3$. This definition will be extended to all dimensions and codimensions in Chapter 3.

The Gauss map. Let $M^2 \subset \mathbb{E}^3$ be oriented and let S^2 denote the unit sphere. Since e_3 is invariant under changes of first-order adapted frame, we obtain a well-defined mapping

$$\begin{aligned} M^2 &\to S^2, \\ x &\mapsto e_3(x), \end{aligned}$$

called the *Gauss map*.

Proposition 2.5.3. $II(v,w) = -\langle de_3(v), w\rangle$.

Thus, II admits the interpretation as the derivative of the Gauss map.

Exercises 2.5.4:

1. Prove Proposition 2.5.3.
2. Let $M^n \subset \mathbb{E}^{n+1}$ be an oriented hypersurface. Define the Gauss map of M and the analogous notions of principal curvatures, mean curvature and Gauss curvature.
3. Show that the generic fibers of the Gauss map are (open subsets of) linear spaces, and thus flat surfaces are ruled by lines. ⊚

A geometric interpretation of Gauss curvature. The round sphere S^2 may be considered as the homogeneous space $ASO(3)/ASO(2)$ via the projection $(x, e_1, e_2, e_3) \mapsto e_3$. As such, the form $\omega^3_1 \wedge \omega^3_2$ is the pullback of the area from on S^2 because $de_3 = -(\omega^3_1 e_1 + \omega^3_2 e_2)$. Since $\omega^3_1 \wedge \omega^3_2 = K\omega^1 \wedge \omega^2$, we may interpret K as a measure of how much the area of M is (infinitesimally) distorted under the Gauss map. (This is because, for a linear map $A : V \to V$, the determinant of A gives, up to sign, the factor by which volume is distorted under A. More precisely, if P is a parallelepiped with one vertex at the origin, $\mathrm{vol}(A(P)) = |\det A|\,\mathrm{vol}(P)$.)

The Gauss equation. There is another way to calculate the Gauss curvature of a surface, namely by differentiating ω^2_1. Using the Maurer-Cartan equation, we obtain

$$d\omega^2_1 = -K\omega^1 \wedge \omega^2, \tag{2.14}$$

which is called the *Gauss equation*.

Exercises 2.5.5:

1. Let $c(t) \subset M^2 \subset \mathbb{E}^3$ be a curve on a surface such that $|c'(t)| = 1$ and $c''(t) \perp T_{c(t)}M$ for all t. Show that $\langle II_{M,c(t)}(c'(t), c'(t)), e_3\rangle = \kappa_{c(t)}$, where κ denotes the curvature of the curve.
2. Let $M^2 \subset \mathbb{E}^3$ be a surface. Let $x \in M$ be a point such that there exists $\epsilon > 0$ such that $M \cap T_xM \cap B_\epsilon(x) = x$, where $B_\epsilon(x)$ denotes a ball in $\mathbb{E}^3$ of radius ϵ around x. Show that $K_x \geq 0$. What can one say if $K_x < 0$?

2.6. Intrinsic and extrinsic geometry

Definition 2.6.1. A *Riemannian manifold* is a differentiable manifold M endowed with a smooth section $g \in \Gamma(M, S^2T^*M)$, called a *Riemannian metric*, that is positive definite at every point.

If $M^n \subset \mathbb{E}^{n+s}$, then the first fundamental form I is a Riemannian metric on M. Which of our differential invariants for M depend only on the induced Riemannian metric I? Such invariants are often called *intrinsic*, depending

only on the Riemannian structure of M, as opposed to *extrinsic* invariants, which depend on how M sits in Euclidean space.

Intrinsic geometry for surfaces. By definition, I is intrinsic, while II is necessarily extrinsic, since it takes values in a bundle that is defined only by virtue of the embedding of M. However, one can obtain intrinsic invariants from II. Given a surface $M^2 \subset \mathbb{E}^3$, we have the "great theorem" of Gauss:

Theorem 2.6.2 (Gauss' theorema egregium). *The Gauss curvature of a surface $M^2 \subset \mathbb{E}^3$ depends only on the induced Riemannian metric.*

Proof. Let $f : U \to M$ be a local parametrization, with $U \subset \mathbb{R}^2$, and let $F : U \to \mathcal{F}^1$ be a first-order adapted lift. Let X_1, X_2 be vector fields on U such that $f_* X_i = e_i$. Then X_1, X_2 are orthonormal for the metric $g = f^* I$ on U. Let η^1, η^2 be the dual 1-forms on U (i.e., $\eta^j(X_i) = \delta^j_i$). Since $\eta^1 \wedge \eta^2 \neq 0$, there exist functions a, b such that

$$\begin{aligned} d\eta^1 &= a\eta^1 \wedge \eta^2, \\ d\eta^2 &= b\eta^1 \wedge \eta^2. \end{aligned}$$

The proof is completed by the following exercises:

Exercises 2.6.3:

1. Show that there exists a 1-form α such that

$$\begin{aligned} d\eta^1 &= -\alpha \wedge \eta^2, \\ d\eta^2 &= \alpha \wedge \eta^1. \end{aligned}$$

2. If $\tilde{X}_1, \tilde{X}_2$ is another g-orthonormal framing on U, show that $d\tilde{\alpha} = d\alpha$. Show that the function κ defined by $d\alpha = \kappa\eta^1 \wedge \eta^2$ is also unchanged, and thus depends only on g.

3. Show that $(\eta^1, \eta^2, \alpha) = F^*(\omega^1, \omega^2, \omega^1_2)$, and thus $\kappa = f^* K$ by (2.14). □

The Codazzi equation. Given two functions H, and K on an open subset $U \subset \mathbb{R}^2$, does there exist (locally) a map $f : U \to \mathbb{E}^3$ such that H and K are the mean and Gauss curvature functions of $M = f(U)$? There are inequalities on admissible pairs of functions because H, K are supposed to be symmetric functions of the principal curvatures (so, e.g., $H = 0$ implies $K \leq 0$). However, as we have already seen in (2.10), stronger restrictions exist and are uncovered when one differentiates and checks that mixed partials commute.

To see these restrictions, set up an EDS on $ASO(3) \times \mathbb{R}^3$, where $\mathbb{R}^3$ has coordinates $h_{ij} = h_{ji}$, for lifts of surfaces equipped with second fundamental forms, namely

$$\mathcal{I} = \{\omega^3, \omega^3_1 - h_{11}\omega^1 - h_{12}\omega^3, \omega^3_2 - h_{21}\omega^1 - h_{22}\omega^3\}_{\mathsf{diff}}$$

with independence condition $\Omega = \omega^1 \wedge \omega^2$. We calculate $d\omega^3 \equiv 0 \operatorname{mod} \mathcal{I}$, but

$$\begin{aligned}
0 &= d\left\{\begin{pmatrix}\omega^3_1\\ \omega^3_2\end{pmatrix} - h\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}\right\}\\
&= -\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^3_1\\ \omega^3_2\end{pmatrix} - dh\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + h\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}\\
&\equiv -\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge h\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} - dh\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + h\begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \operatorname{mod}\mathcal{I}\\
&\equiv -\left\{dh - \left[h, \begin{pmatrix}0 & -\omega^2_1\\ \omega^2_1 & 0\end{pmatrix}\right]\right\}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = 0,
\end{aligned}$$

(where $[,]$ denotes the commutator of matrices). Thus h must satisfy the matrix differential equation (2.6), which is called the *Codazzi equation.*

Given a Riemannian metric g on M and an orthonormal framing, we saw in the proof of Theorem 2.6.2 that $\omega^1, \omega^2, \omega^2_1$ are uniquely determined. In this situation, we may interpret (2.6) as a system of equations for the possible second fundamental forms $II = h_{ij}\omega^i\omega^j$ for embeddings of M into $\mathbb{E}^3$ that induce the metric g. These restrictions are well-defined, since (2.6) is invariant under changes of orthonormal framing.

Intrinsic geometry in higher dimensions.

Frames for any manifold. Given an n-dimensional differentiable manifold M^n, consider the bundle of all *coframings* of M. More precisely, write V for $\mathbb{R}^n$ and let $\pi : \mathcal{F}(M) \to M$ denote the bundle whose fiber over $x \in M$ is the set of all linear maps $f_x : T_xM \to V$. Once we fix a basis of V, we may write a local section of $\mathcal{F}(M)$ as $s(x) = (x, f_x) = (x, \eta^1_x, \ldots, \eta^n_x)$ with $\eta^i \in \Omega^1(M)$ and the η^i_x a basis of T^*_xM. The η^i determine a dual framing $(e_1, \ldots, e_n)$ of TM, so equivalently we may consider $\mathcal{F}(M)$ as the bundle of all *framings* of M.

Although $\mathcal{F}(M)$ is not a Lie group, $GL(V)$ acts simply transitively on the fibers by $g.f_x = g^{-1} \circ f_x$. We now try to obtain an analogue of the Maurer-Cartan form for $\mathcal{F}(M)$:

Define the *tautological* V-valued 1-form η on $\mathcal{F}(M)$ by, for $f = (x, f_x) \in \mathcal{F}(M)$,

$$\eta_f(w) := f_x(\pi_* w), \qquad w \in T_f\mathcal{F}(M).$$

With a basis for V fixed as above, the forms $\pi^*_f(\eta^i)$ (which, by our usual abuse of notation, we write as η^i) furnish a basis of the semi-basic forms on $\mathcal{F}(M)$.

The η^i generalize the semi-basic forms ω^i on the frame bundle of Euclidean space. We would also like to find analogues of the forms ω^i_j, i.e.,

additional forms α^i_j that fill out a coframing of $\mathcal{F}(M)$ and satisfy

$$d\eta^i = -\alpha^i_j \wedge \eta^j. \tag{2.15}$$

Such forms always exist; however, without some additional restrictions, the α^i_j will not be uniquely defined.

Exercise 2.6.4: Suppose $\overline{\eta}^i$ is a local coframing defined on $U \subset M$. Then we may define a local trivialization

$$t : GL(V) \times U \simeq \mathcal{F}(M)|_U$$

by $(g, x) \mapsto g^{-1}\overline{\eta}_x$. Show the existence of a desired coframing on $\mathcal{F}(M)|_U$ as follows:
(a) Show that $t^*\eta = g^{-1}\overline{\eta}$.
(b) Show that there exist $\alpha^i_j \in \Omega^1(\mathcal{F}(M)|_U)$ satisfying (2.15) such that $t^*\alpha^i_j \equiv (g^{-1}dg)^i_j$ modulo $\{\overline{\eta}^i\}$. ⊚
(c) Show that (η^i, α^i_j) is a coframing of $\mathcal{F}(M)|_U$.
(d) Show that any other coframing satisfying (2.15) must be of the form $\tilde{\alpha}^i_j = \alpha^i_j + C^i_{jk}\eta^k$ for some functions $C^i_{jk} = C^i_{kj}$. ⊚

Frames for Riemannian manifolds. Over a Riemannian manifold M, consider the bundle of *orthonormal coframes* which we denote $\pi : \mathcal{F}_{\mathsf{on}}(M) \to M$. Namely, endow V with the standard inner product $\langle , \rangle$ and define the fiber of $\mathcal{F}_{\mathsf{on}}(M) \subset \mathcal{F}(M)$ to be the linear maps $(T_xM, g_x) \to (V, \langle , \rangle)$ that are isometries. (We will sometimes denote the general frame bundle by $\mathcal{F}_{\mathsf{GL}}(M)$ to distinguish it from $\mathcal{F}_{\mathsf{on}}(M)$.) The orthogonal group $O(V)$ acts simply transitively on the fibers of $\mathcal{F}_{\mathsf{on}}(M)$.

On $\mathcal{F}_{\mathsf{on}}(M)$ we still have the tautological forms η^i (which are pullbacks of those on $\mathcal{F}(M)$ under the obvious inclusion). If $s : M \to \mathcal{F}_{\mathsf{on}}(M)$ is a smooth (local) section, then $\overline{\eta}^j = s^*(\eta^j)$ provides a local coframing such that

$$g = (\overline{\eta}^1)^2 + \cdots + (\overline{\eta}^n)^2.$$

Thanks to the additional structure of the Riemannian metric, we will uniquely define an $\mathfrak{so}(V)$-valued 1-form to obtain a canonical framing of $\mathcal{F}_{\mathsf{on}}(M)$ as follows:

Lemma 2.6.5 (The fundamental lemma of Riemannian geometry). *Let (M^n, g) be a Riemannian manifold and let η^i denote the tautological forms on $\mathcal{F}_{\mathsf{on}}(M)$. Let $s : M \to \mathcal{F}_{\mathsf{on}}(M)$ be a smooth section. Then there exist unique forms $\eta^i_j \in \Omega^1(M)$ such that*

i. $s^*(d\eta^i) = -\eta^i_j \wedge s^*(\eta^j)$

and

ii. $\eta^i_j + \eta^j_i = 0.$

Proof. Write $\overline{\eta}^j = s^*(\eta^j)$. Since the $\overline{\eta}^i$ furnish a basis of T^*M at each point, we may write

$$d\overline{\eta}^i = -\alpha^i_j \wedge \overline{\eta}^j \tag{2.16}$$

for some $\alpha^i_j \in \Omega^1(M)$. Let $\alpha^i_j = \beta^i_j + \gamma^i_j$, where $\beta^i_j = -\beta^j_i$ and $\gamma^i_j = \gamma^j_i$.

We first prove existence by showing it is possible to choose forms $\tilde{\alpha}^i_j$ such that $\tilde{\gamma}^i_j = \frac{1}{2}(\tilde{\alpha}^i_j + \tilde{\alpha}^j_i) = 0$.

Exercise 2.6.6: Write $\gamma^i_j = T^i_{jk}\eta^k$ for some functions $T^i_{jk} = T^j_{ik} : M \to \mathbb{R}$, and let

$$\tilde{\alpha}^i_j = \alpha^i_j - (T^i_{jk} + T^i_{kj} - T^j_{ki})\eta^k.$$

Verify that $d\overline{\eta}^i = -\tilde{\alpha}^i_j \wedge \overline{\eta}^j$, and show that $\tilde{\gamma}^i_j = 0$.

To prove uniqueness, assume that α^i_j, $\tilde{\alpha}^i_j$ both satisfy i. and ii. Since $(\tilde{\alpha}^i_j - \alpha^i_j) \wedge \overline{\eta}^j = 0$, by the Cartan Lemma we have $\tilde{\alpha}^i_j - \alpha^i_j = C^i_{jk}\overline{\eta}^k$ for some functions $C^i_{jk} = C^i_{kj}$. Moreover, we also have $C^i_{jk} = -C^j_{ik}$.

Exercise 2.6.7: Show that $C^i_{jk} = 0$.

□

In case M is a submanifold of $\mathbb{E}^{n+s}$, our uniqueness argument implies that $\eta^i_j = F^*(\omega^i_j)$, where F is any extension of s to a first-order adapted framing $F : M \to \mathcal{F}_{\mathbb{E}^{n+s}}$.

In Chapter 8 we will prove the following "upstairs" version of the fundamental lemma:

Lemma 2.6.8 (The fundamental lemma of Riemannian geometry). *Let M be an n-dimensional Riemannian manifold and let η^i denote the tautological forms on $\mathcal{F}_{\mathsf{on}}(M)$. Then there exist unique forms $\eta^i_j \in \Omega^1(\mathcal{F}_{\mathsf{on}}(M))$ such that $d\eta^i = -\eta^i_j \wedge \eta^j$ and $\eta^i_j + \eta^j_i = 0$.*

The η^i_j (in either the upstairs or downstairs version of the fundamental lemma) are referred to as *connection forms.* The upstairs version provides a coframing of $\mathcal{F}_{\mathsf{on}}(M)$, which we now differentiate to obtain differential invariants. (If you are unhappy that we haven't yet proven the upstairs version, you may use the downstairs version and note that all quantities we are about to define are independent of the choice of section s.) While $d\eta^i$ is given by Lemma 2.6.8, we calculate $d\eta^i_j$ by using

$$0 = d^2\eta^i = -(d\eta^i_j + \eta^i_k \wedge \eta^k_j) \wedge \eta^j.$$

Let $\tilde{\Theta}^i_j := d\eta^i_j + \eta^i_k \wedge \eta^k_j \in \Omega^2(\mathcal{F}_{\mathrm{on}}(M))$. The forms $\tilde{\Theta}^i_j$ are semi-basic (because 0 and the η^j are). Define

$$\tilde{\Theta} = \tilde{\Theta}^i_j e^j \otimes e_i \in \Omega^2\left(\mathcal{F}_{\mathrm{on}}(M), \pi^*(\mathrm{End}(TM))\right),$$

where $\mathrm{End}(TM) = T^*M \otimes TM$.

Exercise 2.6.9: Show that $\tilde{\Theta}$ is basic, i.e., show that there exists $\Theta \in \Omega^2(M, \mathrm{End}(TM))$ such that $\tilde{\Theta} = \pi^*(\Theta)$. ⊚

The differential invariant Θ is called the *Riemann curvature tensor.*

Let $\mathfrak{so}(TM) \subset \mathrm{End}(TM)$ denote the subbundle of endomorphisms of TM that are compatible with the Riemannian metric g, in the sense that if $A \in \mathfrak{so}(T_xM)$, then $\rho(A)g_x = 0$, where $\rho : \mathrm{End}(T_xM) \to \mathrm{End}(S^2T^*_xM)$ is the induced action. In other words, if $A \in \mathfrak{so}(T_xM)$, then

$$g_x(A\,v, w) = -g_x(v, A\,w) \quad \forall v, w \in T_xM.$$

Exercise 2.6.10: Show that $\Theta \in \Omega^2(M, \mathfrak{so}(TM))$.

Definition 2.6.11. A Riemannian manifold (M^n, g) is *flat* if there exist local coordinates $x^1, \dots, x^n$ such that $g = (dx^1)^2 + \ldots + (dx^n)^2$. For example, the Riemannian metric on $\mathbb{E}^n$ is flat.

Theorem 2.6.12. *Let (M^n, g) be a Riemannian manifold such that $\Theta \equiv 0$. Then M is flat.*

Proof. By hypothesis, $d\eta^i_j + \eta^i_k \wedge \eta^k_j = 0$. By Cartan's Theorem 1.6.11, around any point $x \in M$ there exist an open set U and a map $g : U \to SO(n)$ such that $\eta^i_j = (g^{-1}dg)^i_j$ on U. We have

$$d\eta^j = -(g^{-1}dg)^j_k \wedge \eta^k.$$

Taking a new frame $\widetilde{\eta}^i = g^i_j \eta^j$, we obtain

$$\begin{aligned} d\widetilde{\eta}^i &= dg^i_j \wedge \eta^j - g^i_j \wedge d\eta^j \\ &= dg^i_j \wedge \eta^j - g^i_j \wedge ((g^{-1}dg)^j_k \wedge \eta^k = 0. \end{aligned}$$

Thus $\widetilde{\eta}^i = dx^i$ for some functions x^i defined on a possibly smaller open set U'. □

The following exercises show how some standard notions from Riemannian geometry are natural consequences of the structure equations described above. If you have not already seen these notions, you may wish to skip these exercises.

Exercises 2.6.13:

Write $\Theta^i_j = \frac{1}{2}R^i_{jkl}\eta^k \wedge \eta^l = \sum_{k>l} R^i_{jkl}\eta^k \wedge \eta^l$, so that

$$\Theta = \tfrac{1}{2}R^i_{jkl}(\eta^j \otimes e_i)\otimes\eta^k \wedge \eta^l \in \Gamma(\mathfrak{so}(TM)\otimes\Lambda^2 T^*M) = \Omega^2(M, \mathfrak{so}(TM)).$$

(Here we should really be pulling everything back from $\mathcal{F}_{\mathsf{on}}(M)$, but we continue to abuse notation and omit the s^*'s.) We can use the Riemannian metric g to define a bundle isomorphism $\sharp : TM \to T^*M$; tensoring with T^*M, we obtain a map $\sharp\otimes\mathrm{Id}_{T^*M} : \mathfrak{so}(TM) \to \Lambda^2(T^*M)$. Applying $\sharp\otimes\mathrm{Id}_{T^*M}$ to the first two factors in Θ, we define

$$R = \tfrac{1}{2}R_{ijkl}(\eta^i \wedge \eta^j)\otimes(\eta^k \wedge \eta^l) \in \Gamma(M, \Lambda^2 T^*M \otimes \Lambda^2 T^*M).$$

1. (a) Show that since we are using orthonormal frames, $R_{ijkl} = R^i_{jkl}$. (If we were using frames that were not orthonormal, then $R_{ijkl} \neq R^i_{jkl}$; see §2.10.)
(b) Show that $R \in \Gamma(S^2(\Lambda^2 T^*M))$, i.e., $R_{ijkl} = R_{klij}$.
(c) Show that $R \in \Gamma(T^*M \otimes S_{21}T^*M))$, i.e., $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. (See Appendix A for the definition of the tensorial construct $S_{21}V$.) This is called the *first Bianchi identity*.
(d) Let $R_{ik} = \sum_j R_{ijkj}$ and $\mathrm{Ric} = R_{ik}\eta^i\eta^k \in \Gamma(S^2T^*M)$. Show that Ric is well-defined. It is called the *Ricci curvature* of M.
(e) Show that $S := \sum_i R_{ii} \in C^\infty(M)$ is well-defined. It is called the *scalar curvature* of M.
(f) Show that when $n = 3$ one can recover R from Ric, but this is not the case for $n > 3$.
(g) Show that when M is a surface, $R_{1212} = K$, the Gauss curvature.
(h) Let $E \in G(2, T_xM)$, the Grassmannian of two-planes in T_xM (see Chapter 3 or Appendix A), and let v_1, v_2 be an orthonormal basis of E. Then we define $K(E) := R(v_1, v_2, v_1, v_2)$ as the *sectional curvature* of E. Show that $K(E)$ is well-defined. (Remark: S, the scalar curvature, satisfies $S(x) = \int_{Gr(2,T_xM)} K(E)\,\mathrm{dvol}$ where dvol is the natural volume form on $Gr(2, T_xM) = SO(T_xM)/(S(O(2) \times S(n-2)))$.
(i) Calculate the sectional curvature function on $G(2, T_x(S^2 \times S^2))$, where $S^2 \times S^2$ has the product metric. What are the maximum and minimum values for $K(E)$?
(j) More generally, given Riemannian manifolds (M_1, g_1) and (M_2, g_2), we can form the product Riemannian manifold $(M_1 \times M_2, g_1 + g_2)$. Express the Riemann curvature tensor of $M_1 \times M_2$ in terms of R_1, R_2, the curvature tensors on M_1, M_2.

For a more invariant description of the various curvatures one can extract from the Riemann curvature tensor, see Exercise A.6.16.1.

2. *(Covariant differential operators)* Let $X \in \Gamma(TM)$ be a vector field and $s : M \to \mathcal{F}_{\mathsf{on}}(M)$ a (local) orthonormal coframing. We have $X = X^i e_i$ for some functions X^i. Define the *covariant derivative* of X to be

$$\nabla X = (dX^i + X^j \eta^i_j) \otimes e_i \in \Omega^1(M, TM) = \Gamma(TM \otimes T^*M), \tag{2.17}$$

with η^i and η^i_j being pulled back via s.
(a) Show that ∇X is well-defined, i.e., independent of the choice of section s.
(b) For $Y \in \Gamma(TM)$, we define $\nabla_Y X := Y \lrcorner \nabla X = (dX^i + X^j \eta^i_j)(Y) e_i$. Show that $\nabla_Y (fX) = f \nabla_Y X + X(f) Y$ for $f \in C^\infty(M)$. In other words, the differential operator ∇_Y obeys the Leibniz rule.
(c) Show that

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.18}$$

Note that the left-hand side, which is the Lie bracket of X and Y, is independent of the Riemannian metric.
(d) Show that ∇ is compatible with g in the sense that

$$Y(g(X_1, X_2)) = g(\nabla_Y X_1, X_2) + g(X_1, \nabla_Y X_2)$$

for all $Y, X_1, X_2 \in \Gamma(TM)$.
(e) If $\alpha = a_i \eta^i \in \Omega^1(M)$, we may define

$$\nabla \alpha = (da_i + a_j \eta^j_i) \otimes \eta^i \Gamma(T^*M \otimes T^*M).$$

Alternatively, define $\nabla \alpha$ by requiring, for all $X, Y \in \Gamma(TM)$, that

$$Y(X \lrcorner \alpha) = (\nabla_Y X) \lrcorner \alpha + X \lrcorner (\nabla_Y \alpha).$$

Since ∇_Y is linear in Y, this defines a tensor $\nabla \alpha \in \Gamma(T^*M \otimes T^*M)$ by

$$\nabla \alpha(X, Y) := X \lrcorner (\nabla_Y \alpha).$$

Show that these two definitions of $\nabla \alpha$ agree. Similarly, one can extend ∇ to act on sections of $T^{*\otimes a} M \otimes T^{\otimes b} M$ and its natural subbundles.

Let M be a C^∞ manifold. A *covariant differential operator* or *connection* on TM is an operator $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that is $C^\infty(M)$-linear in the first factor and obeys the Leibniz rule. If the connection satisfies (2.18), it is called *torsion-free*. The fundamental Lemma 2.6.5 implies that there exists a unique connection that is torsion-free and compatible with the Riemannian metric. This connection is called the *Levi-Civita connection*. Connections are discussed in more detail in §8.2.

3. Let $\nabla R = R_{ijkl,m} (\eta^i \wedge \eta^j) \otimes (\eta^k \wedge \eta^l) \otimes \eta^m$. Show that ∇R satisfies the *second Bianchi identity* $R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0$. ◎

4. For $X, Y, Z \in \Gamma(TM)$, define

$$\overline{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Show that $\overline{R}$ is a tensor, i.e., it is $C^\infty(M)$-linear in all three factors, and that $\overline{R} = -\Theta$. ◎

The fundamental theorem for hypersurfaces. Suppose $M^n \subset \mathbb{E}^{n+s}$ is a submanifold. The relationship between the curvature tensor of the induced Riemannian metric on M and the second fundamental form of M is given by the *algebraic Gauss map* G, defined as follows:

Let V, W be vector spaces, where W has an inner product. Given a basis e^j of V^* and an orthonormal basis f_a of W, let

$$\begin{aligned} G : S^2V^* \otimes W &\to S^2(\Lambda^2 V^*), \\ h^a_{ij} e^i e^j \otimes f_a &\mapsto \sum_a (h^a_{ij} h^a_{kl} - h^a_{il} h^a_{jk})(e^i \wedge e^k) \circ (e^j \wedge e^l), \end{aligned}$$

where $1 \leq i, j, k \leq n = \dim V$ and $1 \leq a, b \leq s = \dim W$. (If we think of each h^a as a matrix, we are taking the 2×2 minors.)

Exercises 2.6.14:

1. Show that G is independent of the choices of bases.
2. Taking $V = T_xM$, $W = N_xM$ (the normal space at x), show that $G(II) = R$.

We are now finally in a position to answer our original question regarding the equivalence of surfaces in $\mathbb{E}^3$, and at the same time see the generalization to hypersurfaces.

Theorem 2.6.15 (The fundamental theorem for hypersurfaces in $\mathbb{E}^{n+1}$). *Let (M, g) be a Riemannian manifold with curvature tensor R, and let $h \in \Gamma(S^2T^*M)$. Assume that*

i. (Gauss) $R = G(h)$

and

ii. (Codazzi) $\nabla h \in \Gamma(S^3T^*M)$

hold. Then for every $x \in M$ there exist an open neighborhood U containing x, and an embedding $f : U \to \mathbb{E}^{n+1}$ as oriented hypersurface, such that $f^*(I) = g$ and $f^*(\langle II, e_{n+1}\rangle) = h$, where e_{n+1} is a unit vector in the direction of the orientation. Moreover, f is unique up to a Euclidean motion.

Corollary 2.6.16. *Let $M^n, \overline{M}^n \subset \mathbb{E}^{n+1}$ be two orientable hypersurfaces with fundamental forms I, II and $\overline{I}, \overline{II}$. Suppose there exist a diffeomorphism $\phi : M \to \overline{M}$ and unit vector fields $e_{n+1}, \overline{e}_{n+1}$ such that*

i. $\phi^*(\overline{I}) = I$,

and

ii. $\phi^*\left(\langle \overline{II}, \bar{e}_{n+1}\rangle\right) = \langle II, e_{n+1}\rangle$.

Then there exists $g \in AO(n+1)$ (the group $ASO(n+s)$ plus reflections) such that $\phi = g\,|_M$.

Proof of 2.6.15. Let $\Sigma = \mathcal{F}_{\mathsf{on}}(M) \times \mathcal{F}_{\mathsf{on}}(\mathbb{E}^{n+1})$, where we use η's to denote forms on the first space and ω's for forms on the second. We omit pullback notation in the proof. Let Ω denote a volume form on $\mathcal{F}_{\mathsf{on}}(M)$ (e.g., wedge together all the entries of the Maurer-Cartan form) and let

$$\mathcal{I} = \{\eta^i - \omega^i, \eta^i_j - \omega^i_j, \omega^{n+1}, \omega^{n+1}_j - h_{jk}\omega^k\}_{\mathsf{diff}}.$$

Integral manifolds of $(\mathcal{I}, \Omega)$ are graphs of immersions $i : \mathcal{F}_{\mathsf{on}}(M)|_U \to \mathcal{F}_{\mathsf{on}}(\mathbb{E}^{n+1})$ that are lifts of immersions $j : U \to \mathbb{E}^{n+1}$, for $U \subset M$, satisfying $j^*(I) = g$ and $j^*(\langle II, e_{n+1}\rangle) = h$. We obtain existence by:

Exercise 2.6.17: Show that $(\mathcal{I}, \Omega)$ is Frobenius. ◎

To prove uniqueness, fix an orthonormal frame $(e_1, \dots, e_n)$ at x. If two such immersions $j, \tilde{\jmath}$ exist, we can arrange (by composing $\tilde{\jmath}$ with an element of $AO(n+1)$) that $j(x) = \tilde{\jmath}(x)$, $j_* e_i = \tilde{\jmath}_* e_i$ at x and the orientations at $j(x)$ match up. Thus, j and $\tilde{\jmath}$ have lifts to Σ which are both integral manifolds of $(\mathcal{I}, \Omega)$ and pass through the same point $(p, q) \in \Sigma$. Thus, $j = \tilde{\jmath}$ by the uniqueness part of the Frobenius Theorem. □

The Laplacian. Let (M^n, g) be a Riemannian manifold with a volume form. Recall the star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ defined in Appendix A. Define a differential operator of order two, the *Laplacian*, by

$$\Delta_g \alpha = (d * d * + * d * d)\alpha, \qquad \alpha \in \Omega^k(M).$$

If $(\omega^1, \dots, \omega^n)$ is a coframing of M, recall the notation $df = f_j \omega^j$.

Exercise 2.6.18: (a) If $M^2 \subset \mathbb{E}^3$ is a surface and (e_1, e_2) is a Darboux framing, with principal curvature functions k_1, k_2, show that

$$\Delta_g f = -(f_{11} + f_{22}) - \frac{f_1 k_{2,1} - f_2 k_{1,2}}{k_1 - k_2}.$$

(b) If g is a flat metric and $(x^1, \dots, x^n)$ are coordinates such that dx^j gives an orthonormal coframing, show that

$$\Delta_g f = -(f_{11} + \dots + f_{nn}).$$

(c) If $x : M^2 \to \mathbb{E}^n$ is an isometric immersion (i.e., the metric g on M agrees with the pullback from $\mathbb{E}^n$), then calculating the Laplacian of each component of x as a vector-valued function gives

$$\Delta_g x = 2\vec{H},$$

where $\vec{H} = \operatorname{trace}_g II$ is the mean curvature vector.

Isothermal coordinates. Let (M^2, g) be a Riemannian manifold with coordinates (x, y). Write $g = a(x,y)dx^2 + b(x,y)dxdy + c(x,y)dy^2$; then one can calculate $K(x, y)$ by differentiating the functions a, b, c. In general one gets a mess (although this was the classical way of calculating K).

Let (M^n, g) be a Riemannian manifold. Coordinates $(x^1, \dots, x^n)$ such that

$$g = e^{2u}(dx^1 \circ dx^1 + \dots + dx^n \circ dx^n),$$

where $u = u(x, y)$ is a given function, are called *isothermal coordinates*. Note that a Riemannian manifold admits isothermal coordinates iff g is conformally equivalent to the flat metric. In Chapter 5 we will show that every surface with an analytic Riemannian metric admits isothermal coordinates. In fact, this is true for C^∞ metrics as well—see ([**142**], vol. IV).

Specializing to surfaces with isothermal coordinates (x, y), the framing $e_1 = e^{-u}\partial x$, $e_2 = e^{-u}\partial y$ is orthonormal.

Exercise 2.6.19: (a) Show that the Gauss curvature is given by

$$K = -e^{-2u}\Delta u,$$

where Δ is the Laplacian. In particular, if $K = \pm 1$, then of course $\Delta u = \mp e^{2u}$. Writing $z = x + iy$, solutions to this are given by

$$u(z) = \log \frac{2|f'(z)|}{1 \pm |f(z)|^2},$$

where f is a holomorphic function on some $D \subset \mathbb{C}$ with $f' \neq 0$ on D and $1 \pm |f|^2 > 0$.
(b) Show that, in isothermal coordinates, $\Delta f = 0$ iff $f_{xx} + f_{yy} = 0$.

2.7. Space forms: the sphere and hyperbolic space

We have seen that $\mathbb{E}^n \cong ASO(n)/SO(n)$ as a homogeneous space. Expressing $\mathbb{E}^n$ in this way facilitated a study of the geometry of its submanifolds.

Let $S^n \subset \mathbb{E}^{n+1}$ be the sphere of radius one, with its inherited metric g. We may similarly express S^n as the quotient $SO(n+1)/SO(n)$. In this manner, $\mathcal{F}_{\mathsf{on}}(S^n) = SO(n+1)$ with the basepoint projection given by $(e_0, e_1, \dots, e_n) \mapsto e_0 \in S^n$.

Let $\mathbb{L}^{n+1}$ be $(n+1)$-dimensional Minkowski space, i.e., $\mathbb{R}^{n+1}$ equipped with a quadratic form

$$Q(x, y) = -x^0y^0 + x^1y^1 + \dots + x^ny^n$$

of signature $(1, n)$. Let $O(V, Q) = O(1, n)$ denote the group of linear transformations preserving Q (see Appendix A for details). Then $\mathbb{L}^{n+1} \cong ASO(1,n)/SO(1,n)$.

We define *hyperbolic space* to be

$$H^n = \{x \in \mathbb{L}^{n+1} \mid Q(x,x) = -1, x^0 > 0\}.$$

(The reasons for this name will become clear below.) Thus, H^n may be considered as (one half of) the "sphere of radius -1" in $\mathbb{L}^{n+1}$.

Exercise 2.7.1: Show that Q restricts to be positive definite on vectors tangent to H^n.

Thus, H^n inherits a Riemannian metric from $\mathbb{L}^{n+1}$. Moreover, H^n can be expressed as the quotient $SO(1,n)/SO(n)$. In this manner, $\mathcal{F}_{\mathsf{on}}(H^n) = SO(1,n)$, with the basepoint projection given by $(e_0, e_1, \ldots, e_n) \mapsto e_0 \in H^n$.

Let $\epsilon = 0, 1, -1$ respectively for $X = \mathbb{E}^n, S^n, H^n$. This handy notation will enable us to study all three spaces and their submanifold geometry at the same time. Then $X = G/SO(n)$, where G is respectively $ASO(n)$, $SO(n+1)$, $SO(1,n)$, with Lie algebra $\mathfrak{g} = \mathfrak{aso}(n)$, $\mathfrak{so}(n+1)$, $\mathfrak{so}(1,n)$. The Maurer-Cartan form of G may be written as

$$\omega = \begin{pmatrix} 0 & -\epsilon\omega^j \\ \omega^i & \omega^i_j \end{pmatrix}$$

where $1 \leq i, j \leq n$ and $\omega^i_j + \omega^j_i = 0$.

As explained above, we identify G with $\mathcal{F}_{\mathsf{on}}(X)$, so that the ω^i are the tautological forms for the projection to X and $g := \Sigma(\omega^\alpha)^2$ gives the Riemannian metric on X. The Maurer-Cartan equation for $d\omega^i$ implies that the ω^i_j are the (upstairs) Levi-Civita connection forms for g. We also use the Maurer-Cartan equation to compute the curvature of X:

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j - \omega^i \wedge (-\epsilon\omega^j).$$

Therefore, $\Theta^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j = \epsilon\omega^i \wedge \omega^j$ and

$$R_{ijkl} = \epsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Exercises 2.7.2:

1. Show that the sectional curvature of X is constant for all 2-planes. (In particular, it is zero for $\mathbb{E}^n$, positive for S^n, and negative for H^n.)
2. Let $M^{n-1} \subset X^n$ be a hypersurface. Define its second fundamental form and describe the hypersurfaces with $II \equiv 0$.
3. Consider the surface $S^1 \times S^1 \subset S^3$ defined by $(x^0)^2 + (x^1)^2 = \cos^2\theta$ and $(x^2)^2 + (x^3)^2 = \sin^2\theta$ for some constant $\theta \in (0, \pi/2)$. Show that this surface, which is known as a *Clifford torus*, is flat, and calculate its mean curvature.

2.8. Curves on surfaces

The interaction between the geometry of surfaces in $\mathbb{E}^3$ and the geometry of curves lying on them was much studied by early differential geometers such

as Dupin, Gauss, Minding and Monge (see [**146**] for more information). We will use curves on surfaces to prove the Gauss-Bonnet theorem and to study Cauchy-type problems associated to the exterior differential systems for surfaces in Chapter 5.

Let $c(s)$ be a regular curve in $\mathbb{E}^3$ parametrized by arclength. Recall from §1.8 that we can adapt frames so that

$$d(x, T, N, B) = (x, T, N, B) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix} ds.$$

Now say that c lies on a surface $M \subset \mathbb{E}^3$. Let (e_1, e_2, e_3) be a first-order adapted lift of M (so $e_3 \perp TM$). Let θ denote the angle from e_1 to T, and let ϵ be T rotated counterclockwise by $\frac{\pi}{2}$ in T_pM, so that

$$\begin{pmatrix} T \\ \epsilon \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then the $\{e_3, \epsilon\}$-plane is orthogonal to T. The angle between N and e_3 is traditionally denoted by ϖ,[1] so that

$$\begin{pmatrix} N \\ B \end{pmatrix} = \begin{pmatrix} \cos\varpi & \sin\varpi \\ -\sin\varpi & \cos\varpi \end{pmatrix} \begin{pmatrix} e_3 \\ \epsilon \end{pmatrix}.$$

Since (T, ϵ, e_3) gives an orthonormal frame of $\mathbb{E}^3$, when we restrict this frame to c we have

$$d(x, T, \epsilon, e_3) = (x, T, \epsilon, e_3) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa_g & \kappa_n \\ 0 & -\kappa_g & 0 & \tau_g \\ 0 & -\kappa_n & -\tau_g & 0 \end{pmatrix} ds \tag{2.19}$$

for some functions $\kappa_g(s), \kappa_n(s), \tau_g(s)$. (This notation will become less mysterious in a moment.)

To interpret these functions, notice that

$\kappa_g = \kappa \sin\varpi =$ component of the orthogonal projection of κN onto ϵ;

$\kappa_n = \kappa \cos\varpi =$ component of the orthogonal projection of κN onto e_3.

The first, κ_g, is called the *geodesic curvature*. *Geodesics* are defined to be the constant speed curves with $\kappa_g \equiv 0$. Exercise 2.8.1.3 below shows that $\kappa_g(c) = \nabla_{c'}c'$, and this shows how geodesic curvature is defined intrinsically for Riemannian manifolds. In Exercise 2.10.9, you will show that geodesics are locally the curves that are the shortest distance between two points on a Riemannian manifold. Thus the notion of a small geodesic disk about a

[1] This letter, pronounced "var-pi" by M. Spivak in our favorite introduction to differential geometry [**142**], is not a sickly omega, but an alternate way of writing π.

point (which we will use in the proof of the Gauss-Bonnet theorem) makes sense.

Next, (2.19) shows that κ_n measures the curving of the surface in the direction of T (by means of measuring how the surface normal e_3 is bending); it is called the *normal curvature* of the surface along c. We say $c \subset M$ is an *asymptotic line* on M if $\kappa_n \equiv 0$.

Finally, notice that if $\kappa_g \equiv 0$, then all of the curvature of the curve lies in the normal direction, and ϵ is parallel to the binormal B of the curve. Thus τ_g measures what the torsion (as a curve in $\mathbb{R}^3$) of a geodesic having tangent vector T would be; it is called the *geodesic torsion* of c.

Exercises 2.8.1:

1. Show that

$$\begin{aligned} \kappa_n &= -\langle II(T,T), e_3\rangle, \\ \tau_g &= \langle II(\epsilon,\epsilon), e_3\rangle, \\ \kappa_g &= (d\theta + \omega_1^2)(T). \end{aligned} \tag{2.20}$$

(Hint: Use Proposition 2.5.3.)

(2.20) shows that κ_g is intrinsic to the induced Riemannian metric on the surface and depends on how the curve is situated on the surface (in particular, how T is turning as we move along c). By contrast, the values of τ_g, κ_n depend only on the pointwise value of T, and are really measuring properties of the immersion of the surface into $\mathbb{E}^3$.

2. Show that $\kappa_g \equiv 0$ if and only if the osculating plane to c is perpendicular to the surface at each point.
3. Show that $\kappa_g(c) = \nabla_{c'}c'$.
4. Find formulas for κ_n, τ_g in terms of the principal curvatures k_1, k_2 when our surface is given a principal (Darboux) framing.
5. Find formulas for κ_n, τ_g when our surface is given a framing such that $e_1 = T$.
6. Calculate τ_g of a curve c such that c' is a principal direction (i.e., a direction where κ_n is a principal curvature) at each point along c. Such curves are called *lines of curvature*.

Exercise 2.8.2: Prove the *local Gauss-Bonnet theorem*: Let (M^2, g) be a Riemannian manifold and let $U \subset M$ be a connected, simply-connected oriented open subset with smooth boundary ∂U. Show that

$$\int_U K dA = 2\pi - \int_{\partial U} \kappa_g ds,$$

where dA is the area form on U, ds is the arclength measure on ∂U, and ∂U is oriented so that if T points along ∂U and N points into the interior of U, then $T \wedge N$ agrees with the orientation on U. ◎

2.9. The Gauss-Bonnet and Poincaré-Hopf theorems

Given a compact oriented surface, there are lots of Riemannian metrics we can put on it. With different metrics, the Gauss curvature can have wildly different behavior. However, as we will see, the integral of the Gauss curvature is independent of the metric and only relies on the underlying topology of M.

First, we review some concepts from simplicial and differential topology.

A *triangulation* of the plane is the plane together with a set of triangular tiles that fill up the plane. A triangulation of a neighborhood U in a surface M is the pullback of some triangulation of a plane under a diffeomorphism $f : U \to \mathbb{R}^2$, and a triangulation of a surface is a covering by triangulated open sets such that the triangulations agree on the overlaps.

Let T be a triangulation of M with V vertices, E edges and F faces. Define

$$\chi_\Delta(M, T) = V - E + F.$$

Exercise 2.9.1: Show that if T, T' are two triangulations of M, then $\chi_\Delta(M, T) = \chi_\Delta(M, T')$. ⊚

Since $\chi_\Delta(M, T)$ is independent of T, we will denote it by $\chi_\Delta(M)$.

Let $X \in \Gamma(TM)$ be a vector field with isolated zeros. Around such a zero $p \in M$ define the *index* of X at p as follows: Pick a closed embedded curve retractible within M to p such that no other zero of X lies in the region U enclosed by the curve. Choose a diffeomorphism $f : U \to D$, where D is the unit disc in $\mathbb{E}^2$. Let θ be the counterclockwise angle between $f_*(X)$ and some fixed vector $v \in \mathbb{E}^2$. Define the integer

$$\mathrm{ind}_X(p) = \frac{1}{2\pi} \int_{\partial D} d\theta,$$

where the circle $\partial D \subset \mathbb{E}^2$ is oriented counterclockwise. The index is well-defined by Stokes' theorem.

Intuitively, to obtain the index we draw a small circle around an isolated zero. Travel around the circle once and count how many times the vector field spins counterclockwise (counting clockwise spin negatively) when going around the circle once. See **[121]** or **[142]** for more on indices of vector fields.

Suppose $p_1, \ldots, p_r \in M$ are the isolated zeros of X. Define

$$\chi_{\mathsf{vf}}(M, X) = \sum_i \mathrm{ind}_X(p_i).$$

Below, we will indirectly prove that $\chi_{\mathsf{vf}}(M, X)$ is independent of X. For a direct proof, again see **[121]** or **[142]**.

Let M^2 be a compact oriented manifold without boundary. Let g be a Riemannian metric on M and define

$$\chi_{\mathsf{metric}}(M,g) = \frac{1}{2\pi}\int_M K dA.$$

Theorem 2.9.2 (Guass-Bonnet and Poincaré-Hopf). *Let (M^2, g) be a compact orientable Riemannian manifold, let T be a triangulation of M and let X be a vector field on M with isolated zeros. Then*

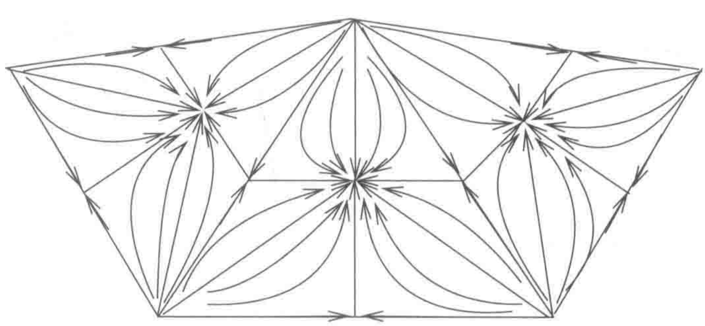
$$\chi_{\mathsf{metric}}(M,g) = \chi_{\Delta}(M) = \chi_{\mathsf{vf}}(M,X).$$

The equality $\chi_{\mathsf{metric}}(M,g) = \chi_{\Delta}(M)$ is the *Gauss-Bonnet Theorem*, and $\chi_{\mathsf{vf}}(M,X) = \chi_{\Delta}(M)$ is the *Poincaré-Hopf Theorem*. The common value of these invariants is called the *Euler characteristic* of M, and is denoted by $\chi(M)$.

We first prove, for certain vector fields X, that $\chi_{\mathsf{vf}}(M,X) = \chi_{\Delta}(M)$. Next we show that $\chi_{\mathsf{metric}}(M,g) = \chi_{\mathsf{vf}}(M,X)$, which, since the left hand side is independent of X and the right hand side independent of g, shows that both are well-defined. Combined with Poincaré-Hopf for certain vector fields, this proves both theorems.

Just for fun, we afterwards give a direct proof that $\chi_{\mathsf{metric}}(M,g) = \chi_{\Delta}(M)$. (Actually, "we" here is a bit of a euphemism, as you, the reader, will do much of the work in Exercise 2.9.3.)

Proof. Given a triangulation T, one can associate a vector field X to it such that $\chi_{\mathsf{vf}}(M,X) = \chi_{\Delta}(M)$. Consider the following picture:



Notice that each vertex of the triangulation becomes a zero of index $+1$, and each edge and face contains a zero of index -1 or $+1$, respectively.

To prove that $\chi_{\mathsf{metric}}(M,g) = \chi_{\Delta}(M)$, we follow ([**142**], vol. III) (who probably followed someone else): We will divide M into two pieces, a subset $U \subset M$ where X is complicated but the topology of U is trivial, and $M \backslash U$ where X is simple but we know nothing about the topology.

Let $p_1, \ldots, p_r$ be the zeros of X. Let $D_i(\epsilon)$ be an open geodesic disc of radius ϵ about p_i, where ϵ is small enough so that the discs are contractible and don't intersect each other. Let $N(\epsilon) = M \backslash (\cup_i D_i(\epsilon))$, a manifold with boundary. On $N(\epsilon)$, X is nonvanishing, so we may define a *global* oriented orthonormal framing on $N(\epsilon)$ by taking $e_1 = \frac{X}{|X|}$ with e_2 determined by the orientation. We let η^1, η^2 denote the dual coframing. Now we calculate

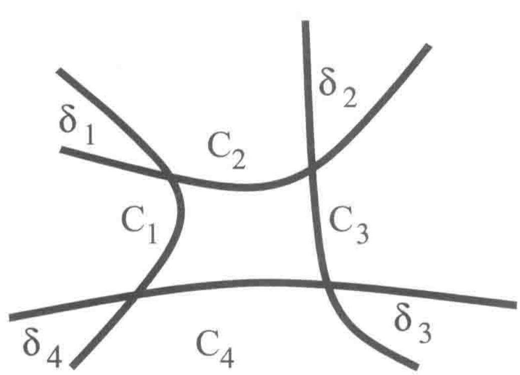
$$\int_{N(\epsilon)} K dA = \int_{N(\epsilon)} d\eta_1^2 = \int_{\partial N(\epsilon)} \eta_1^2 = \sum_i \int_{\partial D_i(\epsilon)} \eta_1^2.$$

Let $\tilde{e}_1^j, \tilde{e}_2^j$ be an orthonormal framing in $D_j(\epsilon)$, and from now on we suppress the j index. Let θ denote the angle between e_1 and $\tilde{e}_1$. We have $\tilde{\eta}_1^2 = \eta_1^2 - d\theta$ wherever both framings are defined. We calculate

$$\begin{aligned}\int_M K dA &= \lim_{\epsilon \to 0} \int_{N(\epsilon)} K dA \\ &= \lim_{\epsilon \to 0} \sum_i \int_{\partial D_i(\epsilon)} \tilde{\eta}_1^2 + d\theta \\ &= \lim_{\epsilon \to 0} \sum_i \int_{D_i(\epsilon)} K dA + \sum_i \int_{\partial D_i(\epsilon)} d\theta. \end{aligned} \tag{2.21}$$

Since K is bounded, as $\epsilon \to 0$ the first expression in (2.21) tends to zero. As ϵ tends to zero the vector $\tilde{e}_1$ tends to a constant vector, so the second term tends towards 2π times index of X at p_i. $\square$

Exercise 2.9.3 (The Gauss-Bonnet Formula)**:** Let (M^2, g) be an oriented Riemannian manifold and let $R \subset M$ be an open subset that is contractible with ∂R the union of a finite number of smooth curves $C_1, \ldots, C_p$, oriented so that if T points along C_j and N points into R, then $T \wedge N$ agrees with the orientation on R. Let δ_i denote the angle between the terminal position of the tangent vector to C_i and the initial position of the tangent vector to C_{i+1} (with the convention that $C_{p+1} = C_1$):



Prove the *Gauss-Bonnet formula:*

$$\int_R K dA = \int_{\partial R} \kappa_g ds + \sum_i \delta_i - 2\pi.$$

Then, use this formula to obtain a second proof of the Gauss-Bonnet theorem using the triangulation definition of the Euler characteristic.

Why are these theorems so wonderful? Take a plane in $\mathbb{E}^3$, and draw a circle in the plane. Now perturb the disk inside the circle—by stretching or squashing, whatever you like—so that the boundary of the disk stays flat (see Figure 1). What is the average curvature of the wildly curving surface

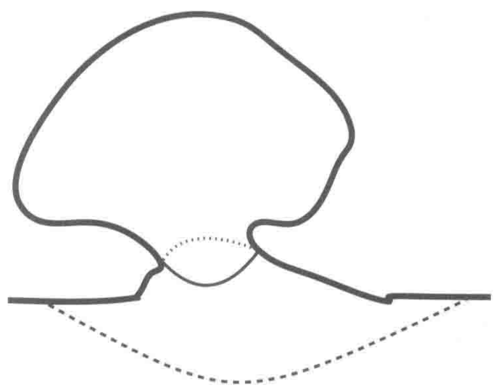


Figure 1. A distorted disk

you've made inside the circle? Zero! Next, take a round sphere, sit on it, twist it, fold it so it gets lots of negative curvature regions. What's the average curvature of your distorted sphere? 4π, no matter how strong you are!

Exercises 2.9.4:

1. Let $M^2 \subset \mathbb{E}^3$. Show that the degree of the Gauss map is $\chi(M)/2$.
2. What is $\int_{\partial R} \kappa_g ds$, where R is the region enclosed by the lower dashed curve (only half of which is pictured) in Figure 2?
3. Prove the Gauss-Bonnet theorem for compact even-dimensional hypersurfaces $M^n \subset \mathbb{E}^{n+1}$. Namely, let K_n denote the product of the principal curvatures $k_1, \ldots, k_n$ and dV the volume element. Then

$$\int_M K_n dV = \frac{1}{2} \operatorname{vol}(S^n)\chi(M),$$

where $\operatorname{vol}(S^n) = 2^{n+1}\pi^{n/2}(\frac{n}{2})!/n!$ is the volume of the n-dimensional unit sphere for n even.

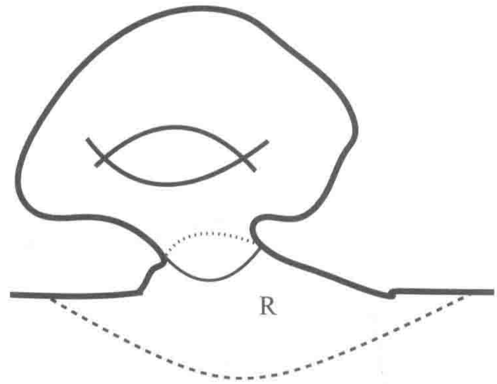


Figure 2. What's the average Gauss curvature?

The generalization by Chern [**34**] to Riemannian manifolds is:

Theorem 2.9.5 (Gauss-Bonnet-Chern). *Let M^n be a compact, oriented Riemannian manifold of even dimension. Then*

$$\int_M \mathrm{Pfaff}(\Theta) dV = 2^{n-1} \operatorname{vol}(S^n) \chi(M),$$

where $\Theta_{ij} = \frac{1}{2} R_{ijkl} \eta^k \wedge \eta^l$.

The Pfaffian is defined in Exercise A.4.8.4. Note that, since the entries of Θ are 2-forms, wedge products are commutative, and so the Pfaffian makes sense with the multiplications being wedge products.

The proof of the Gauss-Bonnet-Chern theorem is not too difficult; see, e.g., [**142**]. The essential point is that if one puts two metrics $g, \tilde{g}$ on M, the forms $\mathrm{Pfaff}(\Theta)dV$ and $\mathrm{Pfaff}(\widetilde{\Theta})dV$ differ by an exact form and therefore $[\mathrm{Pfaff}(\Theta)dV]$ is well-defined as a cohomology class, i.e., $\int_M \mathrm{Pfaff}(\Theta)dV$ is independent of the Riemannian metric. The same proof works for Riemannian metrics on arbitrary vector bundles over M, giving rise to curvature representations of the Euler class of a vector bundle (see [**14**] for the definition of the Euler class). More generally, any elementary symmetric combination of the eigenvalues of a skew-symmetric matrix with, e.g., positive imaginary part leads to a *characteristic class*; that is, the corresponding cohomology class obtained from Θ is independent of the Riemannian metric used. For an excellent introduction to representing characteristic classes via curvature, see the appendix to [**36**].

There are further generalizations of Gauss-Bonnet-Chern (e.g., the Atiyah-Singer index theorem), but discussion of them would take us too far afield at this point; for further reading, see [**113**].

2.10. Non-orthonormal frames

Non-orthonormal frames for Riemannian manifolds. Let M^n be a differentiable manifold and let $\mathcal{F} = \mathcal{F}_{\mathsf{GL}}(M)$ be the bundle of all framings of M, as in §2.6. Suppose M happens to have a Riemannian metric g, but we continue to use $\mathcal{F}(M)$. (This will be desirable if, for example, we wish to vary the metric on M.) We define the functions $g_{ij} = g_{ij}(f) := g(e_i, e_j)$ on $\mathcal{F}$, where $f = (x, e_1, \ldots, e_n) \in \mathcal{F}$. The fundamental lemma of Riemannian geometry now takes the form:

Lemma 2.10.1 (Fundamental Lemma). *There exist unique forms $\eta^i_j \in \Omega^1(\mathcal{F}_{\mathsf{GL}}(M))$ such that*

i. $d\eta^i = -\eta^i_j \wedge \eta^j$

and

ii. $dg_{ij} = g_{ik}\eta^k_j + g_{kj}\eta^k_i$,

where η^i are the tautological forms on $\mathcal{F}_{\mathsf{GL}}(M)$.

Note that the second condition replaces $0 = \eta^i_j + \eta^j_i$, which no longer holds on $\mathcal{F}_{\mathsf{GL}}(M)$.

Exercise 2.10.2: Let (M, g) be oriented and let dvol_M denote the induced volume form. Show that

$$\mathrm{dvol}_M = \sqrt{\det(g_{ij})}\,\eta^1 \wedge \ldots \wedge \eta^n. \quad ⊚$$

Submanifolds of $\mathbb{E}^{n+s}$. Consider the general frame bundle $\mathcal{F}_{\mathsf{GL}}(\mathbb{E}^{n+s})$, which we may identify with $AGL(n+s)$ in the usual way. Let $M^n \subset \mathbb{E}^{n+s}$, and let $\mathcal{F}^1 \subset \mathcal{F}_{\mathsf{GL}}(\mathbb{E}^{n+s})$ be the bundle of first-order adapted frames, i.e., frames such that T_xM is spanned by $e_1, \ldots, e_n$. As a prelude to Chapter 3, define the *quotient normal bundle* $\tilde{N}M$ as the bundle whose fiber at $x \in M$ is $T_x\mathbb{E}^{n+s}/T_xM$. Then $e_{n+1}, \ldots, e_{n+s}$ span $\tilde{N}_xM$ modulo T_xM, but these vectors are not necessarily perpendicular to T_xM.

Using index ranges as in §2.5, on $\mathcal{F}^1$ we have

$$d(x, e_j, e_a) = (x, e_k, e_b) \begin{pmatrix} 0 & 0 & 0 \\ \omega^j & \omega^j_k & \omega^k_b \\ 0 & \omega^a_k & \omega^a_b \end{pmatrix}.$$

Then the Maurer-Cartan equation $0 = d\omega^a = -\omega^a_i \omega^i$ again implies that $\omega^a_i = h^a_{ij}\omega^j$ for some functions $h^a_{ij} = h^a_{ji}$ on $\mathcal{F}^1$, so we have

$$I = g_{ij}\omega^i\omega^j \in \Gamma(M, S^2T^*M),$$
$$II = \omega^a_j\omega^j \otimes e_a \in \Gamma(M, S^2T^*M \otimes \tilde{N}M).$$

Now, for simplicity, assume M is a hypersurface. Fix an orientation on M (say, upward). Let N be an unit vector field perpendicular to the surface, and let $Q = \langle II, N\rangle$. *Then the eigenvalues of $g^{-1}Q$ are well-defined.* This can be explained as follows.

Given a vector space V with a quadratic form $Q \in S^2V^*$, we may think of Q as a map $V \to V^*$. Given a linear map between two different vector spaces, it does not make sense to talk of eigenvalues (and therefore traces and determinants). But now say we have a second, nondegenerate, quadratic form $g \in S^2V^*$. We may think of g^{-1} as a map $g^{-1} : V^* \to V$ and consider the composition $g^{-1} \circ Q : V \to V$. We can calculate the trace and determinant of $g^{-1} \circ Q$.

Exercise 2.10.3: Say M is a surface. Show that

$$\begin{aligned} K &= \det(g^{-1} \circ Q) \\ H &= \operatorname{trace}(g^{-1} \circ Q). \circledcirc \end{aligned}$$

Coordinate formulas for H, K. Now we will finally prove the formulas (1.3). Say $M \subset \mathbb{E}^3$ is given locally by a graph $z = f(x, y)$, with $f(0,0) = 0$ and $f_x(0,0) = f_y(0,0) = 0$.

A simple coframing of $T\mathbb{R}^3$ along M is

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= dy, \\ \omega^3 &= dz - f_x dx - f_y dy. \end{aligned}$$

Note that this coframing is first-order adapted in the sense that $TM = \{\omega^3\}^\perp$. The dual framing is

$$\begin{aligned} e_1 &= \partial_x + f_x \partial_z, \\ e_2 &= \partial_y + f_y \partial_z, \\ e_3 &= \partial_z. \end{aligned}$$

Then

$$(g_{ij}) = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_y f_x & 1 + f_y^2 \end{pmatrix},$$

and therefore

$$\begin{aligned} \mathrm{dvol}_M &= (\det g)^{\frac{1}{2}} \omega^1 \wedge \omega^2 \\ &= [(1 + f_x^2)(1 + f_y)^2) - (f_x f_y)^2]^{\frac{1}{2}} dx \wedge dy \\ &= (1 + f_x^2 + f_y^2)^{\frac{1}{2}} dx \wedge dy. \end{aligned} \tag{2.22}$$

Computing de_1 and de_2 gives $\omega_1^3 = d(f_x)$ and $\omega_2^3 = d(f_x)$, so that $h = (h_{ij})$ is just the Hessian of f.

Exercise 2.10.4: Show that, relative to our framing for the graph $z = f(x, y)$,

$$Q = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$
$$(g_{ij})^{-1} = (1 + f_x^2 + f_y^2)^{-1} \begin{pmatrix} 1 + f_x^2 & -f_x f_y \\ -f_x f_y & 1 + f_y^2 \end{pmatrix}.$$

Then confirm that H, K are given by (1.3). ◎

Geometric interpretation of $H \equiv 0$. Non-orthonormal frames are particularly useful if one wants to deform a submanifold.

Definition 2.10.5. $M^2 \subset \mathbb{E}^3$ is said to be *minimal* if for all $x \in M$ there exists a closed neighborhood U, with $x \in U \subset M$, such that for any $V \subset \mathbb{E}^3$ that is a small deformation of U with $\partial V = \partial U$, we have $\operatorname{area}(V) \geq \operatorname{area}(U)$.

Theorem 2.10.6. *$M^2 \subset \mathbb{E}^3$ is minimal iff $H \equiv 0$.*

Proof. We show that minimal implies $H \equiv 0$. We use our area formula (2.22) and deform the metric. We work locally, so we have $U \subset \mathbb{R}^2$ and $x : U \to \mathbb{E}^3$ giving the surface. Let u, v be coordinates on U.

Fix an orthonormal framing (e_1, e_2, e_3) along $x(U)$ and let $x^t(u, v)$ be a nontrivial deformation of x. For t sufficiently small, we may write

$$x^t(u, v) = x(u, v) + t\, s(u, v) e_3(u, v) + O(t^2),$$

where $s : U \to \mathbb{R}$ is some function. For fixed t, we calculate

$$\begin{aligned} dx^t &= dx + t\, e_3\, ds + t\, s\, de_3 + O(t^2) \\ &= e_1\omega^1 + e_2\omega^2 + t\, e_3(s_1\omega^1 + s_2\omega^2) - t\, s(e_1\omega_1^3 + e_2\omega_2^3) + O(t^2). \end{aligned}$$

Using the same ω^1, ω^2, we may write $dx^t = e_1^t\omega^1 + e_2^t\omega^2$ for

$$\begin{aligned} e_1^t &= (1 - t\, s\, h_{11})e_1 - t(s\, h_{21} e_2 + s_1 e_3) + O(t^2), \\ e_2^t &= (1 - t\, s\, h_{22})e_2 - t(s\, h_{12} e_1 + s_2 e_3) + O(t^2). \end{aligned}$$

Of course, this framing is no longer orthonormal, and in fact the metric now looks like

$$g^t = \begin{pmatrix} 1 - t\, s\, h_{11} & 2t\, s\, h_{21} \\ 2t\, s\, h_{12} & 1 - t\, s\, h_{22} \end{pmatrix} + O(t^2).$$

Exercise 2.10.7: Calculate $\frac{d}{dt}|_{t=0}(\det g^t)^{\frac{1}{2}}$ and show $\frac{d}{dt}|_{t=0} \int_U \operatorname{dvol}(g^t) = 0$ iff $H \equiv 0$.

Now we wave our hands a little and ask you to trust that calculus in infinite dimensions behaves the same way as in finite dimensions. That is, a function has a critical point at a point where the derivative vanishes, and in our case its easy to see we are at a minimum. (If you don't believe us, consult a rigorous book on the calculus of variations, such as [**59**].) □

Exercise 2.10.8: More generally, for $M^n \subset \mathbb{E}^{n+s}$, define $\vec{H} \in \Gamma(NM)$, the *mean curvature vector*, to be $\operatorname{trace}_g(II)$. Show that M is minimal iff $\vec{H} \equiv 0$. In particular, show that straight lines are locally the shortest curves between two points in the plane

Exercise 2.10.9: More generally, let X^{n+s} be a Riemannian manifold and $M^n \subset X$ a submanifold, and define $\vec{H} \in \Gamma(NM)$, the *mean curvature vector*, to be $\operatorname{trace}_g(II)$. Show that M is minimal iff $\vec{H} \equiv 0$. In particular, show that geodesics are locally the shortest curves between two points of X.

Chapter 3

Projective Geometry

This chapter may be considered as an update to the paper of Griffiths and Harris [**69**], which began a synthesis of modern algebraic geometry and moving frames techniques. Other than the first three sections, it may be skipped by readers anxious to arrive at the Cartan-Kähler Theorem. An earlier version of this chapter, containing more algebraic results than presented here, constituted the monograph [**107**].

We study the local geometry of submanifolds of projective space and applications to algebraic geometry. We begin in §3.1 with a discussion of Grassmannians, one of the most important classes of manifolds in all of geometry, and some uses of Grassmannians in Euclidean geometry. We define the Euclidean and projective Gauss maps. We then describe moving frames for submanifolds of projective space and define the projective second fundamental form in §3.2. In §3.3 we give some basic definitions from algebraic geometry. We give examples of homogeneous algebraic varieties and explain several constructions of auxiliary varieties from a given variety $X \subset \mathbb{P}V$: the secant variety $\sigma(X)$, the tangential variety $\tau(X)$ and the dual variety X^*. In §3.4 we describe the basic properties of varieties with degenerate Gauss maps and classify the surface case. We return in §§3.5–3.7 to discuss moving frames and differential invariants in more detail, with plenty of homogeneous examples in §3.6. We discuss osculating hypersurfaces and prove higher-order Bertini theorems in §3.7.

In §3.8 and §3.9, we apply our machinery respectively to study uniruled varieties and to characterize quadric hypersurfaces (Fubini's Theorem). Varieties with degenerate duals and associated varieties are discussed in §3.10 and §3.11 respectively. We prove the bounds of Zak and Landman on the dual defect from our differential-geometric perspective. In §3.12 we study

varieties with degenerate Gauss images in further detail. In §3.14 we state and prove *rank restriction theorems*: we show that the projective second fundamental form has certain genericity properties in small codimension if X is not too singular. We describe how to calculate $\dim \sigma(X)$ and $\dim \tau(X)$ infinitesimally, and state the Fulton-Hansen Theorem relating tangential and secant varieties. In §3.13 we state Zak's theorem classifying *Severi varieties*, the smooth varieties of minimal codimension having secant defects. Section §3.15 is dedicated to the proof of Zak's theorem. In §3.16 we generalize Fubini's Theorem to higher codimension, and finally in §3.17 we discuss applications to the study of complete intersections.

In this chapter, when we work over the complex numbers, all tangent, cotangent, etc., spaces are the holomorphic tangent, cotangent, etc., spaces (see Appendix C). We will generally use X to denote an algebraic variety and M to denote a complex manifold.

Throughout this chapter we often commit the following abuse of notation: We omit the $\circ$ in symmetric products and the $\otimes$ when the product is clear from the context. For example, we will often write $\omega_0^\alpha e_\alpha$ for $\omega_0^\alpha \otimes e_\alpha$.

3.1. Grassmannians

In projective geometry, Grassmannians play a central role, so we begin with a study of Grassmannians and the Plücker embedding. We also give applications to Euclidean geometry.

We fix index ranges $1 \leq i, j \leq k$, and $k+1 \leq s, t, u \leq n$ for this section.

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $G(k, V)$ denote the *Grassmannian* of k-planes that pass through the origin in V. To specify a k-plane E, it is sufficient to specify a basis $v_1, \dots, v_k$ of E. We continue our notational convention that $\{v_1, \dots, v_k\}$ denotes the span of the vectors $v_1, \dots, v_k$. After fixing a reference basis, we identify $GL(V)$ with the set of bases for V, and define a map

$$\begin{aligned} \pi : GL(V) &\to G(k, V), \\ (e_1, \dots, e_n) &\mapsto \{e_1, \dots, e_k\}, \end{aligned}$$

If we let $\tilde{e}_1, \dots, \tilde{e}_n$ denote the standard basis of V, i.e., $\tilde{e}_A$ is a column vector with a 1 in the A-th slot and zeros elsewhere, the fiber of this mapping over $\pi(\mathrm{Id}) = \{\tilde{e}_1, \dots, \tilde{e}_k\}$, is the subgroup

$$P_k = \left\{ g = \begin{pmatrix} g_j^i & g_s^i \\ 0 & g_s^t \end{pmatrix} \mid \det(g) \neq 0 \right\} \subset GL(V).$$

More generally, for $g \in GL(V)$, $\pi^{-1}(\pi(g)) = g P_k g^{-1}$.

Of particular importance is *projective space* $\mathbb{P}V = G(1, V)$, the space of all lines through the origin in V. We define a *line* in $\mathbb{P}V$ to be the

projectivization of a two-dimensional linear subspace of V. (While we won't deal with it here, one can put natural metrics on $\mathbb{P}V$, called the *Fubini-Study metric* in the case V is complex, see **[68]**.)

We may consider $G(k,V)$ as the space of all $\mathbb{P}^{k-1}$'s in $\mathbb{P}V$.

Notation. If $Y \subset \mathbb{P}V$, we let $\hat{Y} \subset V$ denote its pre-image under the projection $V \backslash 0 \to \mathbb{P}V$, called the *cone over* Y. If $Z \subset V$, we let $[Z] = \pi(Z)$.

Exercises 3.1.1:

1. Show there is a canonical isomorphism $G(k,V) \simeq G(n-k, V^*)$. In particular, $\mathbb{P}V^*$ is the space of hyperplanes (i.e., $\mathbb{P}^{n-1}$'s) in $\mathbb{P}V$.
2. Show that the following map, called the *Plücker embedding* of the Grassmannian, is well-defined:

$$G(k,V) \to \mathbb{P}(\Lambda^k V),$$
$$\{e_1, \ldots, e_k\} \mapsto [e_1 \wedge \ldots \wedge e_k]. \circledcirc$$

Proposition 3.1.2. *Let $E \in G(k,V)$. There is a canonical identification*

$$T_E G(k,V) \simeq E^* \otimes (V/E).$$

In particular, $\dim G(k,V) = k(\dim V - k)$.

Remark 3.1.3. The identification $T_E G(k,V) \simeq E^* \otimes (V/E)$ coincides with the intuition that a k-plane near a point E can be described as the graph of a linear map from E to V/E.

First proof. Let $C = \{E(t) = [e_1(t) \wedge \ldots \wedge e_k(t)]\} \subset G(k,V)$ denote a curve in $G(k,V)$ with $E(0) = E = [e_1 \wedge \ldots \wedge e_k]$. (We have $t \in \mathbb{R}$ or $t \in \mathbb{C}$, depending on our ground field.) It will be easier to differentiate in the vector space $\Lambda^k V$, so let $\vec{E}(t) = e_1(t) \wedge \ldots \wedge e_k(t) \in \hat{G}(k,V) \subset \Lambda^k V$. Then

$$\vec{E}'(0) = \sum_{l=1}^{k} e_1 \wedge \ldots \wedge e_{l-1} \wedge e_l'(0) \wedge e_{l+1} \wedge \ldots \wedge e_k.$$

We consider the term corresponding to a fixed l: If $e_l'(0) \in \{e_1, \ldots, e_{l-1}, e_{l+1}, \ldots, e_k\}$, the term is zero. If $e_l'(0)$ is a multiple of e_l, then the term in $\vec{E}'(0)$ containing $e_l'(0)$ is a scalar multiple of $\vec{E}(0)$, and thus $E'(0)$ will be zero iff $\vec{E}'(0)$ is a scalar multiple of $\vec{E}(0)$.

Consider $e_l(t) : \mathbb{C} \to V$. By identifying the tangent space to a point of a vector space with the vector space itself, the differential $de_l \mid_{t_0}: T_{t_0}\{e_l(t_0)\} \to T_{e_l(t_0)}V$ may be considered as a map $de_l \mid_{t_0}: \{e_l(t_0)\} \to V$, i.e., $de_l \in \{e_l\}^* \otimes V$. However we are only interested in $de_l \bmod E$, so we obtain an element of $\{e_l\}^* \otimes V/E$. Summing over l, we obtain $T_E G(k,V) \simeq E^* \otimes (V/E)$. It is clear that the result is independent of our choice of basis for E. $\square$

We do some preliminary work before our second proof. Write the Maurer-Cartan form of $GL(V)$ as

$$\omega = \begin{pmatrix} \omega^i_j & \omega^i_t \\ \omega^s_j & \omega^s_t \end{pmatrix}.$$

Proposition 3.1.4. *The forms ω^s_i in the Maurer-Cartan form of $GL(V)$ are semi-basic for the projection to $G(k, V)$.*

Proof. A curve $c(t) = (e_1(t), \dots, e_n(t)) \subset GL(V)$ is vertical for the projection $\pi : GL(V) \to G(k, V)$ if and only if

$$e_j(t)' \subset \{e_1(t), \dots, e_k(t)\}$$

for all $1 \leq j \leq k$. But $de_j = e_i\omega^i_j + e_s\omega^s_j$, and thus on a curve $c(t)$,

$$\frac{de_j}{dt} = e_i\omega^i_j(c'(t)) + e_s\omega^s_j(c'(t)).$$

So, $c(t)$ is vertical iff $\omega^s_j(c'(t)) = 0$ for all s, j. $\square$

Second proof of 3.1.2. Let (e^j, e^s) be the dual basis to (e_i, e_t). Write $E = \{e_j\}$. For $f = (e_1, \dots, e_n) \in GL(V)$, consider the tensor

$$L_f := \omega^s_i \otimes e^i \otimes (e_s \bmod E) \in T^*_f GL(V) \otimes (E^* \otimes V/E).$$

In Exercise 3.1.5 you will show that L is basic for the projection π, so L descends to a well-defined linear map $T_E G(k, V) \to E^* \otimes V/E$, which is an isomorphism because the ω^s_i are semi-basic and independent, and the two vector spaces have the same dimension. $\square$

Exercise 3.1.5: Show that L is indeed basic, i.e., that it descends to a well-defined element of $T^*_E G(k, V) \otimes (E^* \otimes V/E)$, by showing that the actions of P_k on the three terms cancel each other.

Third proof of 3.1.2. Recall from Chapter 1 that $T_{\pi(\mathrm{Id})} G/P \simeq \mathfrak{g}/\mathfrak{p}$.

In the case of $G = GL(V)$, we have $\mathfrak{g} = V^* \otimes V$. Let $E_0 = \pi(\mathrm{Id}) \in G(k, V)$; then $\mathfrak{p} \simeq (E_0^* \otimes E_0) \oplus ((V/E_0)^* \otimes E_0) \oplus ((V/E_0)^* \otimes (V/E_0))$. (If this confuses you, look at the block form of P_k above.) Writing $V^* \otimes V = (E_0^* + (V/E_0)^*) \otimes (E_0 + V/E_0)$, we obtain $\mathfrak{g}/\mathfrak{p} \simeq E_0^* \otimes (V/E_0)$. In fact we may drop the subscript 0, because the tangent space at $g \in GL(V)$ of the fiber over $E = \pi(g)$ is $L_{g*}\mathfrak{p}$, and the conjugate of $\mathfrak{g} = \mathfrak{gl}(V)$ is still isomorphic to $V^* \otimes V$. $\square$

Note that in the very special case of $\mathbb{P}V = G(1, V)$, we have $T_x\mathbb{P}V = \hat{x}^* \otimes (V/\hat{x})$.

Exercise 3.1.6: Verify that the three different definitions of the isomorphisms $T_E G(k, V) \simeq E^* \otimes (V/E)$ coincide.

Orthogonal Grassmannians and the Euclidean Gauss map. In this subsection only, V denotes a real vector space.

If V is endowed with an inner product and orientation, we define the (orthogonal) Grassmannian $Gr(k,V) = SO(n)/S(O(k)\times O(n-k))$, where

$$\begin{aligned} &S(O(k)\times O(n-k)) \\ &\quad = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \, A \in O(k),\ B \in O(n-k),\ \det(A)\det(B) = 1 \right\}. \end{aligned}$$

Exercises 3.1.7:

1. Show that, as differentiable manifolds, $Gr(k,V) = G(k,V)$.
2. Show that $Gr(k,V)$ inherits additional structure from the metric on V, including a Riemannian metric such that under any local section $s : Gr(k,V) \to SO(V)$, the forms $s^*(\omega^s_j)$ provide an orthonormal coframing. Calculate the Riemann curvature tensor for this metric.
3. Calculate the induced Riemann curvature tensor on $Gr(k,V)$ via its Plücker embedding and compare it to the one above. (Note that $\Lambda^k V$ has an inner product on it induced from the one on V which induces a Riemannian metric on $\mathbb{P}\Lambda^k V$.) Calculate the sectional curvature function $K : G(2, T_pG(k,V)) \to \mathbb{R}$ (see §2.6).
4. Show that there is a canonical identification of the tangent space to $Gr(k,V)$ at E with the space of linear maps $E \to E^\perp$.
5. Show that there is a canonical identification $Gr(k,V) \simeq Gr(n-k,V)$.

The Euclidean Gauss Map.

Definition 3.1.8. We define the *Gauss map* of a submanifold $M^n \subset \mathbb{E}^{n+s}$ as follows: Let $V = T_0\mathbb{E}^{n+s}$ and define

$$\gamma : M \to Gr(n,V)$$

by mapping $x \in M$ to the translate of T_xM to the origin.

Exercises 3.1.9:

1. Let $M^n \subset \mathbb{E}^{n+s}$. Show that $d\gamma = II$, the Euclidean second fundamental form. (Hint: use the identification in 3.1.7.4.) Thus if $c(t)$ is a curve on M, then we may interpret $II(c',c')$ as measuring how c' is moving away from $T_{c(t)}M$ infinitesimally. In particular, suppose M contains a line l. Show that if $x \in l$, then $II(v,v)_x = 0$ if v is a tangent vector to l.
2. Let $M^2 \subset \mathbb{E}^3$ be a surface and let $x \in M$. Show that the Gauss curvature at x may be described as follows:

$$K_x = \lim_{U \to x} \frac{\operatorname{area}(\gamma(U))}{\operatorname{area}(U)},$$

where U is an open neighborhood of x and the area in the numerator is from the induced metric in the Grassmannian $Gr(2,3) = S^2$ and the area in the denominator is calculated in $\mathbb{E}^3$.

3. Let g_0 denote the induced metric on $S^2 = Gr(2,3) \subset \mathbb{E}^3$. Let $M^2 \subset \mathbb{E}^3$. Calculate $\gamma^*(g_0)$ in terms of the metric on M and the coefficients of II.
4. In what follows $M^2 \subset \mathbb{E}^3$ is a surface.
(a) If $\dim \gamma(M) = 1$, then the mean curvature H is not identically zero.
(b) If there exists a line $l \subset M$, then $K_x \leq 0$ for $x \in l$. If there exist at least two distinct lines $l_1, l_2 \subset M$ and $x \in (l_1 \cap l_2) \subset M$ and $K_x < 0$, then either the lines are orthogonal or $H_x \neq 0$. If there exist at least three lines passing through x and contained in M, then $K_x = H_x = 0$.
5. Calculate the images of the Gauss maps of the surfaces of revolution (see §2.2).
6. Let M be the hyperboloid $x^2 + y^2 = 1 + z^2$. Calculate the equation of a line l on M passing through the point $(1,0,0)$. Calculate $\gamma(l)$, where γ is the Gauss map of M.
7. Let $M^2 \subset \mathbb{E}^3$ be a surface with $H = 0$. Let g_γ denote the Riemannian metric induced on M via the pullback of the Gauss map. Show that $g_\gamma = K((\omega_0^1)^2 + (\omega_0^2)^2)$. Thus in this case, γ is a *conformal* mapping, i.e., angles are preserved: $\angle(v,w) = \angle(d\gamma_x(v), d\gamma_x(w))$ for all $v, w \in T_xM$. For another proof, see §6.4.

3.2. Frames and the projective second fundamental form

In this section we set up moving frames for submanifolds of projective space and define the projective second fundamental form. Throughout this section $V = \mathbb{C}^{n+a+1}$ or $\mathbb{R}^{n+a+1}$, and $M \subset \mathbb{P}V$ is accordingly an n-dimensional complex or real submanifold.

Tangent spaces and the projective Gauss map. Let $v \in \hat{M} \subset V$ and let $x = [v] \in M$. Define

$$\hat{T}_xM := T_v\hat{M} \subset V, \text{ the } \textit{affine tangent space},$$
$$\text{and}$$
$$\tilde{T}_xM := \mathbb{P}(\hat{T}_vM) \subset \mathbb{P}V, \text{ the } \textit{embedded tangent (projective) space}.$$

Note that $T_v\hat{M} = T_w\hat{M}$ for all nonzero $v, w \in \hat{x}$, so $\hat{T}_xM, \tilde{T}_xM$ are well-defined. For any submanifold of V, the affine tangent space is the naïve tangent space obtained by translating tangent vectors to the origin using the identification $T_vV \simeq V$. The affine and embedded tangent spaces should not be confused with the (intrinsic) tangent space T_xM defined in Appendix B. Recall from §3.1 that the intrinsic tangent space to $\mathbb{P}V$ has the description $T_x\mathbb{P}V \simeq \hat{x}^* \otimes (V/\hat{x})$.

Exercise 3.2.1: Show that $T_xM \simeq \hat{x}^* \otimes (\hat{T}_xM/\hat{x})$. Find the corresponding description of T_x^*M.

Remark 3.2.2. Readers familiar with notation from algebraic geometry should note that the line bundle with fiber $\hat{x}$ at $x \in M$ is $\mathcal{O}_M(-1)$.

Define the *normal space* $N_xM := T_x\mathbb{P}V/T_xM$. Note that $N_xM = \hat{x}^*\otimes (V/\hat{T}_xM)$. Define the *conormal space* N^*_xM to be the dual vector space, and note that $N^*_xM = \hat{x}\otimes(\hat{T}_xM)^\perp$ is such that $N^*_xM\otimes\mathcal{O}_M(1)_x \subset V^*$. While the normal space is merely a quotient space, the projectivized conormal space determines a linear subspace of $\mathbb{P}V^*$.

Exercise 3.2.3: Show that the linear subspace of $\mathbb{P}V^*$ determined by N^*_xM admits the geometric interpretation as the space of hyperplanes containing $\tilde{T}_xM$.

Define the (projective) *Gauss map*

$$\begin{aligned} \gamma : M &\to G(n+1, V), \\ x &\mapsto \hat{T}_xM. \end{aligned}$$

We call $\gamma(M)$ the *Gauss image* of M. Consider the derivative

$$d\gamma_x : T_xM \to T_{\hat{T}_xM}G(n+1, V),$$

which tells us how M is moving away from its embedded tangent space to first order.

Proposition 3.2.4. *For all $v \in T_xM$, $d\gamma_x(v) \in \operatorname{Hom}(\hat{T}_xM, V/\hat{T}_xM)$ is such that for all $w \in \hat{x}$, $w \in \ker(d\gamma_x(v))$. Thus we have an induced map*

$$\underline{d\gamma_x}(v) \in \operatorname{Hom}((\hat{T}_xM/\hat{x}), V/\hat{T}_xM),$$

i.e., $\underline{d\gamma_x} \in T^*_xM\otimes T^*_xM\otimes N_xM$. Moreover, $\underline{d\gamma_x} \in S^2T^*_xM\otimes N_xM$.

We (or more precisely, you) prove Proposition 3.2.4 in Exercises 3.2.7 and 3.2.8.

Definition 3.2.5. Define $II_{M,x} = \underline{d\gamma_x} \in S^2T^*_xM\otimes N_xM$, the *projective second fundamental form of M at x.*

Remark 3.2.6. Recall from Appendix A that, given a tensor $A \in V^*\otimes W$, we may consider A as a linear map, with either $A : V \to W$ or $A : W^* \to V^*$. In standard linear algebra texts, if one map is given first, the other is denoted ${}^tA$. In order to avoid prejudicing ourselves, we will use A to denote both maps. In particular, we will write both $II : N^*_xM \to S^2T^*_xM$ and $II : S^2T_xM \to N_xM$ when using II.

Frames. We continue with the projection

$$\begin{aligned} \pi : GL(V) &\to \mathbb{P}V, \\ (e_0, \ldots, e_{n+a}) &\mapsto [e_0]. \end{aligned}$$

We fix a reference basis of V as in §3.1 so that we may identify $GL(V)$ with $\mathcal{F}_{\mathbb{P}V}$, the bundle of all framings of $\mathbb{P}V$. Given $M^n \subset \mathbb{P}V$, let $\mathcal{F}^0_M := \mathcal{F}_{\mathbb{P}V}\mid_{\pi^{-1}(M)}$, the 0th-order adapted frames to M.

The affine tangent space of M at each $p \in M$ determines a flag

$$\hat{p} \subset \hat{T}_p M \subset V.$$

Define $\mathcal{F}^1 = \mathcal{F}^1_M \to M$, the *bundle of first-order adapted frames*, to be the frames respecting this flag, namely,

$$\mathcal{F}^1_M = \{(e_0, \dots, e_{n+a}) \in \mathcal{F}^0_M \mid [e_0] \in M,\ \hat{T}_{[e_0]}M = \{e_0, \dots, e_n\}\}.$$

Consequently, on $\mathcal{F}^1 = \mathcal{F}^1_M$, $de_0 \equiv 0 \operatorname{mod}\{e_0, \dots, e_n\}$. In other words, $\frac{de_0(t)}{dt}\mid_{t=0} \in \{e_0, \dots, e_n\}$ for all curves $[e_0(t)] \in M$ with $e_0(0) = e_0$.

The fiber of $\pi : \mathcal{F}^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g \in GL(V) \mid g = \begin{pmatrix} g^0_0 & g^0_\beta & g^0_\nu \\ 0 & g^\alpha_\beta & g^\alpha_\nu \\ 0 & 0 & g^\mu_\nu \end{pmatrix} \right\},$$

where we use the index ranges $1 \le \alpha, \beta \le n$, and $n+1 \le \mu, \nu \le n+a$. Write the pullback of the Maurer-Cartan form to $\mathcal{F}^1$ as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu \\ \omega^\alpha_0 & \omega^\alpha_\beta & \omega^\alpha_\nu \\ \omega^\mu_0 & \omega^\mu_\beta & \omega^\mu_\nu \end{pmatrix}.$$

Since $de_0 = e_0\omega^0_0 + e_1\omega^1_0 + \ldots + e_{n+a}\omega^{n+a}_0$, and $de_0 \equiv 0 \operatorname{mod}\{e_0, \dots, e_n\}$ when pulled back to $\mathcal{F}^1$, we must have $\omega^\mu_0 = 0\ \forall \mu$.

Notation. Given $f = (e_0, \dots, e_{n+a}) \in \mathcal{F}^1$, we let $(e^0, \dots, e^{n+a})$ denote the dual basis of V^* and $\underline{e}_\alpha = e^0 \otimes (e_\alpha \operatorname{mod} e_0) \in T_{[e_0]}M$ denote the tangent vector corresponding to e_α (see Exercise 3.2.1). Note that for any section $s : M \to \mathcal{F}^1$, we have $s^*(\omega^\alpha_0)(\underline{e}_\beta) = \delta^\alpha_\beta$. Also note that $\underline{e}_\alpha$ depends on f and not just e_α. Similarly, for normal vectors, we let $\underline{e}_\mu = e^0 \otimes (e_\mu \operatorname{mod} \hat{T}_{[e_0]}M) \in N_{[e_0]}M$.

Note that we may recover the Gauss map γ from the map

$$\begin{aligned} \mathcal{F}^1 &\to G(n+1, V), \\ f &\mapsto [e_0 \wedge \cdots \wedge e_n]. \end{aligned}$$

Exercise 3.2.7: Prove the first assertion in Proposition 3.2.4 by calculating $d(e_0 \wedge \cdots \wedge e_n)$.

We have our usual Pavlovian response to seeing something equal to zero (and our usual abuse of notation omitting pullbacks in the notation here

and in what follows), so we expand out

$$0 = d\omega_0^\mu = -\omega_\alpha^\mu \wedge \omega_0^\alpha.$$

The Cartan Lemma A.1.9 implies

$$\omega_\beta^\mu = q_{\alpha\beta}^\mu \omega_0^\alpha$$

for some functions $q_{\alpha\beta}^\mu = q_{\beta\alpha}^\mu$ defined on $\mathcal{F}^1$.

Exercise 3.2.8: Prove the "moreover" assertion in Proposition 3.2.4.

Frame definition of the projective second fundamental form. While our next definition of the projective second fundamental form is less elegant, the technique we use to arrive at it is applicable to more general situations, e.g., the definition of the cubic form in §3.5.

Say we did not know about the Gauss map, but we calculated $\omega_0^\mu = 0$ and $d\omega_0^\mu = 0$ on $\mathcal{F}^1$ to obtain functions $q_{\alpha\beta}^\mu : \mathcal{F}^1 \to \mathbb{C}$. We would then like to form an invariant tensor from these functions. To do so, we need to see how they vary as we move in the fiber of $\pi : \mathcal{F}^1 \to M$. Let $\tilde{f} = (\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\nu)$ be another point in $\pi^{-1}([e_0])$. Then there exists $g = (g_B^A) \in P_k$ such that

$$\begin{aligned}
\tilde{e}_0 &= g_0^0 e_0, \\
\tilde{e}_\alpha &= g_\alpha^0 e_0 + g_\alpha^\beta e_\beta, \\
\tilde{e}_\nu &= g_\mu^0 e_0 + g_\mu^\beta e_\beta + g_\mu^\nu e_\nu.
\end{aligned}$$

We first determine how $\omega_0^\alpha, \omega_\alpha^\mu$ vary. Let "$g.$" denote the action of G both on V and on the Maurer-Cartan form, and observe that

$$\begin{aligned}
g.de_0 &= g.(e_0\omega_0^0 + e_\alpha\omega_0^\alpha) \\
&= (g_0^0 e_0)(g.\omega_0^0) + (g_\alpha^0 e_0 + g_\alpha^\beta e_\beta)(g.\omega_0^\alpha).
\end{aligned}$$

On the other hand, since d commutes with pullback,

$$\begin{aligned}
g.de_0 &= d(e_0 g_0^0) \\
&= g_0^0(e_0\omega_0^0 + e_\alpha\omega_0^\alpha),
\end{aligned}$$

the first line holding because d commutes with the g-action. Let h denote the inverse matrix to g. The analogous calculation for de_α combined with our calculation above yields

$$\begin{aligned}
\tilde{\omega}_0^\alpha &= h_\beta^\alpha g_0^0 \omega_0^\beta, \\
\tilde{\omega}_\alpha^\mu &= g_\alpha^\beta h_\nu^\mu \omega_\beta^\nu.
\end{aligned}$$

Substituting into the equation $\tilde{\omega}_\beta^\mu = \tilde{q}_{\beta\alpha}^\mu \tilde{\omega}^\alpha$, we obtain

$$\tilde{q}_{\alpha\beta}^\mu = h_0^0 h_\beta^\gamma h_\alpha^\delta g_\nu^\mu q_{\gamma\delta}^\nu.$$

Thus the quantity

$$\widetilde{II} = q^{\mu}_{\alpha\beta}\omega^{\alpha}_{0}\omega^{\beta}_{0} \otimes \underline{e}_{\mu} \in \pi^*(S^2 T^*_{[e_0]} M \otimes N_{[e_0]} M)$$

is constant on the fiber and therefore descends to be a well-defined section of $S^2T^*M \otimes NM$, which is the projective second fundamental form.

Remark 3.2.9. When we compare the projective second fundamental form to its Euclidean counterpart, unlike the Euclidean case we no longer have a notion of 'how fast' M is moving away from its embedded tangent space to first order in each direction, but only whether or not it is moving away. For example, if M is a complex hypersurface, the only numerical information in the projective second fundamental form is its rank because the only invariant of a quadratic form over $\mathbb{C}$ is its rank. For real projective hypersurfaces the only numerical information is rank and signature.

Remark 3.2.10. If we are in the holomorphic category, we may compare the projective second fundamental form with its Hermitian counterpart. In this case the projective second fundamental form is simply the holomorphic component of the Hermitian second fundamental form.

Remark 3.2.11. The projective second fundamental form admits the following interpretation. As remarked in Exercise 3.2.3, the conormal space N^*_xM may be identified (after tensoring with a line bundle) with the space of linear forms on V annihilating $\hat{T}_xM$, so we may think of $\vec{H} \in N^*_xM$ as determining a hyperplane H containing the tangent projective space $\tilde{T}_xM$. As such, as long as $M \not\subseteq H$, $M \cap H$ must be singular at x because it is defined by the equations of M and the linear form defining H, but this linear form has differential in the span of the differentials of the defining equations of M. With this perspective $II(\vec{H}) \in S^2T^*_xM$ may be identified as the quadratic part of the singularity.

We will use the notation

$$|II_{M,x}| = II(N^*_xM) \subset S^2T^*_xM$$

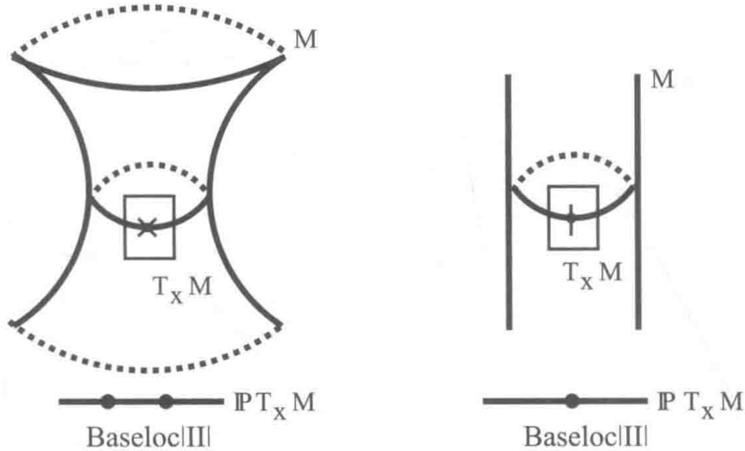
to study the space of quadratic polynomials on T_xM determined by II.

The second fundamental form determines preferred tangent directions. Let

$$\begin{aligned} &\operatorname{Baseloc}|II_{M,x}| = \mathbb{P}\{v \in T_xM \mid II(v,v) = 0\}, \\ &\operatorname{Singloc}|II_{M,x}| = \{v \in T_xM \mid II(v,w) = 0 \ \forall w \in T_xM\}. \end{aligned} \tag{3.1}$$

Baseloc$(II_{M,x})$ is often called the set of *asymptotic directions*, while the dimension of Singloc$(II_{M,x})$ is called the *index of relative nullity*. Note that Baseloc$(II_{M,x})$ is an intersection of quadric (i.e., degree two) hypersurfaces in $\mathbb{P}T_xM$ and that Singloc$(II_{M,x})$ is a linear space.

Note that $\mathbb{P}\,\mathrm{Singloc}(II_{M,x}) \subseteq \mathrm{Baseloc}(II_{M,x})$. For the projectivization of a cylinder in affine 3-space we have equality, while for the projectivization of a hyperbola in affine space, $\mathrm{Baseloc}(II_{M,x})$ consists of two points while $\mathbb{P}\,\mathrm{Singloc}(II_{M,x})$ is empty.



If we think of $\mathbb{P}T_xM \subset \mathbb{P}(T_x\mathbb{P}V)$ as the set of directions where there exists a line having contact to order one with M at x, then $\mathrm{Baseloc}(II_{M,x}) \subset \mathbb{P}T_xM$ is the set of directions where there exists a line having contact to order two with M at x. We may interpret $\mathrm{Singloc}(II_{M,x})$ as $\ker d\gamma_x$, so manifolds where $\mathrm{Singloc}(II_{M,x}) \neq 0$ at general points have degenerate Gauss maps. We discuss this further in §3.4.

Exercises 3.2.12:

1. Show that $\mathrm{Singloc}\,|II| = \{v \in T_xM \mid (s^*\omega^\mu_\alpha)(v) = 0,\ \forall \alpha, \mu\}$, where $s : M \to \mathcal{F}^1$ is any section.
2. Show that if $II_{M,x} \equiv 0$ for all $x \in M$, then M is (an open subset of) a linear $\mathbb{P}^n \subset \mathbb{P}V$. In particular, if M is analytic, then M is an open subset of a linear space if $II_{M,x} \equiv 0$ at a general point $x \in M$.

Remark 3.2.13. If M is given in local linear coordinates as a graph $x^\mu = f^\mu(x^1, \dots, x^n)$, then

$$II_{M,x} = \frac{\partial^2 f^\mu}{\partial x^\alpha \partial x^\beta}(x) dx^\alpha \circ dx^\beta \otimes \frac{\partial}{\partial x^\mu}.$$

See §3.5 for the proof.

3.3. Algebraic varieties

In this section we briefly cover some basic definitions from algebraic geometry to enable us to study global properties of submanifolds of projective space and make initial connections with algebraic geometry. We also give

examples of homogeneous varieties and describe, given a variety $X \subset \mathbb{P}V$, new varieties constructed from X: tangential varieties, secant varieties, and dual varieties. We will use coarse properties of the new varieties to determine properties of the original variety. For example, in §3.4, we will see that if $0 < \dim \gamma(X) < \dim X$, then X cannot be a smooth variety.

Even if our original interest is only in complex manifolds, we will be forced to deal with algebraic varieties. For example, the Gauss image $\gamma(M)$ of a compact complex submanifold of $\mathbb{P}V$ (other than a linear space or quadric hypersurface) is never a complex manifold but it is an algebraic variety.

Definitions. As a working definition, an *algebraic variety* is the projectivization (i.e., the image under $\pi : V\backslash 0 \to \mathbb{P}V$) of the common zero locus of a collection of homogeneous polynomials on V. The *ideal* $I(X) \subset S^{\bullet}V^*$ of a variety $X \subset \mathbb{P}V$ is the set of all homogeneous polynomials vanishing on X. We let $I_k(X) = I(X) \cap S^kV^*$.

By *Chow's Theorem* (see, e.g., [**122**]), any compact complex submanifold of a projective space is an algebraic variety. In the other direction, any algebraic variety, minus at most a proper subvariety, is an analytic manifold. The subvariety consists of points where the Jacobian of the local defining equations drops rank; these are called the *singular* points of X, and we denote the subvariety by X_{sing}.

We define the *dimension* of X to be the dimension of the corresponding complex manifold $X_{\mathsf{smooth}} = X\backslash X_{\mathsf{sing}}$. (If X has several components, we must define the dimension of each component.) A basic property of projective space is that if $X, Y \subset \mathbb{P}V$ are respectively subvarieties of codimension a, b, then $\operatorname{codim}(X \cap Y) \leq a + b$, with equality holding if they intersect transversely.

The *degree* of X is the maximum number of points of intersection with a linear space of complementary dimension. If X is a hypersurface defined by an equation $\{f = 0\}$, then the degree of X is the degree of the polynomial f. *Bezout's Theorem* (see, e.g., [**122**]) states that if $X, Y \subset \mathbb{P}V$ are two varieties of complementary dimension that intersect transversely, then $X \cap Y$ consists of $\deg(X)\deg(Y)$ points.

A discussion of the Zariski topology and rational maps would take us too far afield here, but readers familiar with such terminology should note that the proper language for discussing Gauss maps and their higher-order analogs is that of rational maps. For example, we could either define $\gamma(X)$ as the closure of $\gamma(X_{\mathsf{smooth}})$ or consider γ as a rational map and define $\gamma(X)$ as its strict transform. Sometimes in statements of theorems we refer to a Zariski open subset. The reader may omit the word Zariski without

any harm, because Zariski open subsets of algebraic varieties always contain open subsets in the manifold topology.

We will also have need of the terms *generic* and *general*. Given a vector space V with additional structure S, e.g., a collection of tensors or an ideal of polynomials on V, we define a point $v \in V$ to be *S-generic* if it is generic with respect to S. For example, if V is equipped with a quadratic form Q, $v \in V$ is Q-generic if $Q(v, v) \neq 0$, and more generally, if V is equipped with a space S of quadratic forms, v is S-generic if all small perturbations of v have similar behavior with respect to S. In particular, all integer valued invariants obtainable from v and S are locally constant if one varies v. If the structure S is understood, we omit the quantifier S. If $X \subset \mathbb{P}V$ is a variety, $x \in X$ is a general point of X if all $\mathbb{Z}$-valued differential invariants of X (e.g., the dimension of $N_{2,x}X$) are locally constant in some neighborhood of X. For example, if one considers the plane curve $y = x^3$, then all points on the curve are general except the origin which is a *flex point* (also called a *point of inflection*). The general points of X form a non-empty Zariski open subset of X, as do the S-generic points of V (for any S).

We will say a curve c has *contact to order d with X at x* if there exists a curve $\tilde{c} \subset X$ such that the Taylor series of the curves $c, \tilde{c}$ agree to order d at x. For those familiar with the definition of multiplicity in algebraic geometry, a curve has contact to order d at x if the multiplicity of $c \cap X$ at x is $(d+1)$.

Exercise 3.3.1: Let X be a hypersurface of degree d. Show that any line intersecting X in $d+1$ points is contained in X. Similarly, show that if a line has contact to order d at any point $x \in X$, then it is contained in X. Show that the same conclusion holds if X may be described as the intersection of hypersurfaces of degree at most d. ⊚

Proposition 3.3.2. *If $X \subset \mathbb{P}V$ is a variety defined by polynomials of degree at most two, then* $\operatorname{Baseloc}(II_{X,x})$ *is the set of tangent directions to lines contained in X that pass through x.*

Proof. By Exercise 3.3.1 any line having contact to order two with X at x is contained in X. □

Homogeneous algebraic varieties.

Definition 3.3.3. A subvariety $X \subset \mathbb{P}V$ is said to be *projective rational homogeneous* (or *homogeneous* for short) if X is the orbit in $\mathbb{P}V$ of a point under the action of a subgroup $G \subset GL(V)$. In this case we write $X = G/P$, where P is the isotropy subgroup.

Remark 3.3.4. A projective rational homogeneous variety is always the projectivization of the orbit of a highest weight vector in V, see, e.g., [**52**].

The group P is called a *parabolic* subgroup. An unpublished theorem of Kostant states that if $X = G/P \subset \mathbb{P}V_\lambda$ is a rational homogeneous variety of the G-module V_λ of highest weight λ (see Appendix A), then the ideal of X is generated in degree two by $V_{2\lambda}{}^\perp \subset S^2V^*$.

Remark 3.3.5. Borel and Remmert proved that any compact homogeneous Kähler manifold is of the form $G/P \times T$, where T is the quotient of $\mathbb{C}^g$ by a lattice; see [**2**].

The Grassmannians $G(k, W) \subset \mathbb{P}(\Lambda^k W)$ are homogeneous algebraic varieties; see [**72**] for an explicit description of their ideals. For example, if $W = \mathbb{C}^m$ and we identify $\Lambda^2 W$ with the space of skew-symmetric $m \times m$ matrices, then $G(2, W) \subset \mathbb{P}(\Lambda^2 W)$ may be thought of as the projectivization of the space of rank two skew matrices. Thus $G(2, W)$ may be defined by the 3×3 minors. Kostant's result above implies it is generated in degree two, not three. In fact, the ideal of $G(2, W)$ is generated by the Pfaffians (square roots of the determinants, see Exercise A.4.8.4) of the 4×4 minors centered along the diagonal.

Segre varieties. Let $W = \mathbb{C}^m$ and $U = \mathbb{C}^n$. The *Segre variety* $\mathrm{Seg}(\mathbb{P}W \times \mathbb{P}U) \subset \mathbb{P}(W \otimes U)$ is defined to be the projectivization of the decomposable tensors, i.e., the image of the map

$$\mathrm{Seg} : \mathbb{P}U \times \mathbb{P}W \to \mathbb{P}(U \otimes W),$$
$$([x], [y]) \mapsto [x \otimes y].$$

It is homogeneous for the $G = GL(U) \times GL(W)$ action on $U \otimes W$. If we identify $U \otimes W$ with the space of $n \times m$ matrices, $\mathrm{Seg}(\mathbb{P}W \times \mathbb{P}U)$ is the set of rank one matrices. (Recall that every rank one matrix is the product of a column vector with a row vector.) Thus, its ideal is generated by the 2×2 minors. To have a 2×2 minor we need to choose two rows and two columns, so $I_2(\mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W)) \simeq \Lambda^2 U^* \otimes \Lambda^2 W^*$ and I_2 generates the ideal.

If $X \subset \mathbb{P}U$ and $Y \subset B\mathbb{P}W$ are varieties, we define the *Segre product* of X and Y, $\mathrm{Seg}(X \times Y) \subset \mathbb{P}U \otimes \mathbb{P}W$, to be the image of $X \times Y$ under the map Seg.

Exercises 3.3.6:

1. Show that $I_2(G(2, W)) \simeq \Lambda^4 W^*$.
2. Let $M_r \subset \mathbb{P}(U \otimes W)$ denote the projectivization of the set of matrices of rank at most r. Show that M_r is a variety, that its ideal is generated by polynomials of degree $r + 1$, and $I_{r+1}(M_r) \simeq \Lambda^{r+1} U^* \otimes \Lambda^{r+1} W^*$.

Isotropic Grassmannians. Let $Q \in S^2V^*$, $\omega \in \Lambda^2 V^*$ be nondegenerate. Define the corresponding isotropic Grassmannians

$$G_{Q\text{-null}}(k, V) := \{E \in G(k, V) \mid Q(v, w) = 0 \ \forall v, w \in E\},$$
$$G_{\omega\text{-null}}(k, V) := \{E \in G(k, V) \mid \omega(v, w) = 0 \ \forall v, w \in E\}.$$

In the case $\dim V = 2m$ and $k = m$, $G_{Q-\mathsf{null}}(m, 2m)$ has two isomorphic components. The components are called the *spinor varieties* $\mathbb{S}_m$. The isotropic Grassmannians are naturally varieties, as they sit inside Grassmannians and the isotropy conditions are defined by polynomials. They are homogeneous for $O(V, Q)$ and $Sp(V, \omega)$ respectively.

Veronese varieties. Let S^2V denote the symmetric matrices and let $v_2(\mathbb{P}V) \subset \mathbb{P}(S^2V)$ denote the projectivization of the rank one elements. Then $v_2(\mathbb{P}V)$ is the image of $\mathbb{P}V$ under the injective mapping

$$\begin{aligned} v_2 : \mathbb{P}V &\to \mathbb{P}S^2V, \\ [v] &\mapsto [v \circ v]. \end{aligned}$$

The *d-th Veronese embedding of* $\mathbb{P}V$, $v_d(\mathbb{P}V) \subset \mathbb{P}S^dV$, is defined by $v_d([v]) = [v^d] = [v\circ \ldots \circ v]$. Intrinsically, all these varieties are $\mathbb{P}V$, but the embeddings are projectively inequivalent. We may interpret $v_d(\mathbb{P}V) \subset \mathbb{P}S^dV$ as the set of degree d polynomials on $\mathbb{P}V^*$ whose zero set is a hyperplane counted with multiplicity d.

Given $X \subset \mathbb{P}V$, we can consider the *Veronese re-embeddings of* X, $v_d(X) \subset \mathbb{P}(S^dV)$, which are the restrictions to X of the Veronese embeddings of $\mathbb{P}V$. These will turn out to be useful in our study of osculating hypersurfaces and complete intersections.

Exercise 3.3.7: If Z is a hypersurface of degree d, show that $v_d(Z) = H_Z \cap v_d(\mathbb{P}V)$, where H_Z is the hyperplane in $\mathbb{P}(S^dV)$ annihilated by the equation of Z (which is an element of S^dV^*).

Flag varieties. Let $V = \mathbb{C}^n$ and let $1 \le a_1 < a_2 < \ldots < a_p \le n-1$. A *(partial) flag* of V is a sequence of linear subspaces $0 \subset E_1 \subset E_2 \subset \cdots \subset E_p$ with $\dim E_j = a_j$. Let $\mathbb{F}_{a_1,\ldots,a_p}(V)$ denote the space of all such flags. Note that $G(k, V) = \mathbb{F}_k(V)$. At the other extreme is the variety of *complete flags* $\mathbb{F}_{1,2,\ldots,n-1}(V)$. The flag varieties are clearly homogeneous spaces of $SL(V)$.

Exercises 3.3.8:

1. Write $\mathbb{F}_{1,2,\ldots,n-1} = SL(n)/B$ and explicitly determine B as a matrix Lie group. If $\mathbb{F}_{a_1,\ldots,a_p} = SL(n)/P$, what is P as a matrix Lie group?
2. Determine $T_{[\mathrm{Id}]}\mathbb{F}_{a_1,\ldots,a_p}$. That is, write $E_j = \{e_0, \ldots, e_{j-1}\}$ and generalize our $T_EG(k, V) = E^* \otimes V/E$ description.
3. Show that the flag variety $\mathbb{F}_{1,n}(\mathbb{C}^{n+1})$ can be realized as the space of rank one and traceless matrices.

The flag varieties embed as homogeneous subvarieties of projective space. One way to see this is to consider the Segre product of Grassmannians:

$$S = \mathrm{Seg}(G(a_1, V) \times G(a_2, V) \times \cdots \times G(a_p, V)) \subset \mathbb{P}(\Lambda^{a_1}V \otimes \cdots \otimes \Lambda^{a_p}V)$$

and now to consider the subvariety defined by *incidence relations*

$$\mathbb{F} = \{(E_1, \dots, E_p) \in S \mid E_1 \subset E_2, E_2 \subset E_3, \dots, E_{p-1} \subset E_p\},$$

and observe that $\mathbb{F} = \mathbb{F}_{a_1,\dots,a_p}(V)$. The incidence relations are described by polynomials (namely various contractions being zero).

Some constructions of new varieties from old.

Tangential varieties. Let $M^n \subset \mathbb{P}V = \mathbb{P}^{n+a}$ be a smooth variety. Define $\tau(M) \subset \mathbb{P}V$ by

$$\tau(M) := \bigcup_{x \in M} \tilde{T}_x M,$$

the *tangential variety* of M.

Notice that $\dim \tau(M) \leq \min\{2n, n+a\}$. We will see that for most smooth varieties equality holds. The tangential variety is indeed an algebraic variety, see [**72**].

It is possible to define the tangential variety of an algebraic variety that is not necessarily smooth. While there are several possible definitions, the best one appears to be the union of *tangent stars* $T^\star_x X$. This is because the Fulton-Hansen Theorem 3.13.7 applies with this definition. Intuitively, $T^\star_x X$ is the limit of secant lines. More precisely, let $x \in X$. Then $\mathbb{P}^1_*$ is a line in $T^\star_x X$ if there exist smooth curves $p(t), q(t)$ on X such that $p(0) = q(0) = x$ and $\mathbb{P}^1_* = \lim_{t\to 0} \mathbb{P}^1_{p(t)q(t)}$, where $\mathbb{P}^1_{pq}$ denotes the projective line through p and q. $T^\star_x X$ is the union of all $\mathbb{P}^1_*$'s at x. In general $\overline{\tau(X_{\mathsf{smooth}})} \subseteq \tau(X)$, and strict containment is possible.

Exercises 3.3.9:

1. Let $M \subset \mathbb{P}V$ be a smooth curve that is not a line. Show that $\tau(M)$ has dimension two.
2. Let X be a variety and let $x \in X_{\mathsf{smooth}}$. Show that $T^\star_x X = \tilde{T}_x X$.

Joins, cones and secant varieties. Let $Y, Z \subset \mathbb{P}V$ be varieties (we allow the possibility that $Y = Z$). For $x, y \in \mathbb{P}V$, let $\mathbb{P}^1_{xy}$ denote the projective line containing x and y. Define the *join* of Y and Z to be

$$J(Y, Z) = \overline{\bigcup_{x\in Y, y\in Z, x\neq y} \mathbb{P}^1_{xy}}.$$

If $Z = \mathbb{P}^k$ is a k-plane, we call $J(Y, \mathbb{P}^k)$ the *cone* over Y with vertex $\mathbb{P}^k$. If $Y = Z$, we call $\sigma(Y) = J(Y, Y)$ the *secant variety* of Y. We may similarly define the join of k varieties to be the union of the corresponding $\mathbb{P}^{k-1}$'s, or by induction as $J(Y_1, \dots, Y_k) = J(Y_1, J(Y_2, \dots, Y_k))$. In particular, we let $\sigma_k(Y) = J(Y, \dots, Y)$ denote the join of k copies of Y, called the *k-th secant variety of Y*.

Aside 3.3.10. Given $Y \subset \mathbb{P}V$, if Y is not contained in a hyperplane we obtain a stratification of $\mathbb{P}V$ by the secant varieties of Y. Given $x \in \mathbb{P}V$,

define the *essential rank* of x to be the minimum k such that $x \in \sigma_k(Y)$ and the *rank* of x to be the minimum k such that x is in the span of k points of Y. The notions of rank and essential rank provide a geometric generalization of the corresponding notions for tensors (and in particular matrices).

Lemma 3.3.11 (Terracini's Lemma). *If $[x] \in J(Y,Z)_{\mathsf{smooth}}$ with $[x] = [y+z]$, such that $[y] \in Y_{\mathsf{smooth}}$, $[z] \in Z_{\mathsf{smooth}}$, then*

$$\hat{T}_{[x]}J(Y,Z) = \hat{T}_{[y]}Y + \hat{T}_{[z]}Z.$$

Proof. Consider the addition map add : $V \times V \to V$ given by $(v,w) \mapsto v+w$. Then

$$\hat{J}(Y,Z) = \overline{\mathrm{add}(\hat{Y} \times \hat{Z})}.$$

Now consider the differential of add $|_{\hat{Y}\times\hat{Z}}$ at a general point. The result follows. □

Note that we expect $\dim J(Y,Z) = \dim Y + \dim Z + 1$, because, given $[y+z] \in J(Y,Z)$, we are free to move y in $\dim Y$ directions, z in $\dim Z$ directions, and we may move along the line joining our points. The following exercises discuss this expectation.

Exercises 3.3.12:

1. Show that if there exist $y \in Y$ and $z \in Z$ such that $\tilde{T}_yY \cap \tilde{T}_zZ = \emptyset$, then $\dim J(Y,Z) = \dim Y + \dim Z + 1$.
2. Suppose $X \subset \mathbb{P}V$ is a curve not contained in a plane. Show that $\dim \sigma(X) = 3$.
3. Let $X = \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$. Calculate $\dim \sigma(X)$.

Dual varieties.

Definition 3.3.13. Let $X \subset \mathbb{P}V$ be a variety. Define

$$X^* := \overline{\{H \in \mathbb{P}V^* \mid \exists x \in X_{\mathsf{smooth}} \text{ such that } \tilde{T}_xX \subseteq H\}} \subset \mathbb{P}V^*,$$

the *dual variety* of X.

Note that X^* coincides with $\gamma(X)$ when X is a hypersurface.

Intuitively, for $X^n \subset \mathbb{P}^{n+a}$, we expect X^* to be a hypersurface, as there are n dimensions worth of points on X and there is an $(a-1)$-dimensional space of hyperplanes tangent to each smooth point of X.

Exercises 3.3.14:

1. Show that if X is a curve that is not contained in a hyperplane, then X^* is a hypersurface.
2. Show that if X is a cone over a linear space, then X^* is not a hypersurface.

Linear projections and hyperplane sections. Given $X \subset \mathbb{P}V$ and $p \in \mathbb{P}V \backslash X$, we consider the vector space projection $V \to V/\hat{p}$. The image of $\hat{X}$ under this projection is still a cone, and when we projectivize it, we obtain a subvariety $X' \subset \mathbb{P}(V/\hat{p})$, called the *projection of X from p*.

Similarly, given a variety $X \subset \mathbb{P}V$ and a hyperplane $H \subset \mathbb{P}V$, we define the *hyperplane section* of X by H to be $X \cap H$. Hyperplane sections and linear projections are related in Exercise 3.3.15.4 below.

Exercises 3.3.15:

1. Assume $p \notin \tilde{T}_x X$. Calculate $II_{X',x'}$ in terms of $II_{X,x}$. ⊚
2. Let $H \subset \mathbb{P}V$ be a hyperplane and assume $\tilde{T}_x X \not\subset H$. Calculate $II_{X \cap H,x}$ in terms of $II_{X,x}$. ⊚
3. Show that the degree of $X \cap H$ is the same as that of X.
4. Show that if X is not a hypersurface, then $(X')^* = X^* \cap H_p$, where $H_p \subset \mathbb{P}V^*$ is the hyperplane corresponding to the point $p \in \mathbb{P}V$.

Incidence correspondences. We have already seen that flag varieties can be described as *incidence varieties* of Segre varieties, that is, a natural subvariety of a product of varieties. Many important constructions will use incidence varieties, and the auxiliary varieties defined above are naturally defined using incidence varieties. For example, one can study Gauss images and dual varieties using incidence varieties as follows:

Let $X^n \subset \mathbb{P}V$ be a variety. Consider

$$\Gamma(X) = \overline{\{(x, \tilde{T}_x X) \mid x \in X_{\mathsf{smooth}}\}} \subset \mathbb{F}_{1,n+1}(V) \tag{3.2}$$

with its two projections:

$$\begin{tikzcd} & \Gamma(X) \arrow[dl] \arrow[dr] & \\ X & & G(n+1, V). \end{tikzcd}$$

Similarly consider

$$\mathcal{I}_{X,X^*} = \overline{\{(x, H) \mid x \in X_{\mathsf{smooth}}, \tilde{T}_x X \subset H\}} \subset \mathbb{F}_{1,n+a}(V) \tag{3.3}$$

with picture

$$\begin{tikzcd} & \mathcal{I} \arrow[dl] \arrow[dr] & \\ X & & X^*. \end{tikzcd}$$

One advantage of using incidence correspondences is that they guide us to a natural frame bundle over X for studying the problem at hand. For example, when studying dual varieties, we will work with a frame bundle that factors through the first-order adapted frame bundle of $\mathcal{I}_{X,X^*}$. This method will be used consistently throughout this chapter.

3.4. Varieties with degenerate Gauss mappings

In this section we begin our study of varieties $X \subset \mathbb{P}V$ such that $\dim \gamma(X) < \dim X$. We show such varieties are always singular and locally ruled by the fibers of the Gauss map, which are linear spaces. (A variety is *locally ruled* by fibers isomorphic to F if an open subset $U \subset X$ is a fibration, with fibers (Zariski) open subsets of F.) We classify surfaces with degenerate Gauss images and characterize subvarieties of the Grassmannian that occur as Gauss images. The study of varieties with degenerate Gauss images is continued in §3.12.

Proposition 3.4.1. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Let $x \in X$ be a general point, let F be the component of $\gamma^{-1}(\gamma(x))$ containing x and let $f = \dim F$. Then*

1. *$T_xF = \mathrm{Singloc}\,|II_{X,x}|$,*
2. *F is a linear space, and thus X is locally ruled by f-planes.*

Conversely, if $X \subset \mathbb{P}V$ is a variety, $x \in X$ a general point and $\mathrm{Singloc}(II_{X,x})$ is nonempty, then γ is degenerate with fiber equal to the linear space F such that $T_xF = \mathrm{Singloc}\,|II_{X,x}|$.

This proposition is essentially due to C. Segre; see ([**137**], p. 95), where assertion 2 is explicitly proven.

Moreover, we have

Theorem 3.4.2. *Let $X^n \subset \mathbb{P}^{n+a}$ be a subvariety. If X is not a linear subspace of $\mathbb{P}^{n+a}$ and $\dim \gamma(X) < \dim X$, then X is singular. More precisely, if F is a general fiber of γ, then X_{sing} contains a codimension one subset of F which is the zero locus of a polynomial of degree $n-f$ on $\hat{F}$.*

The subset of $X_{\mathsf{sing}} \cap F$ referred to in the theorem is called the *focal hypersurface of X along F* and will be denoted by Φ.

Proof of 3.4.1. Consider the commutative diagram

$$\begin{tikzcd} & \mathcal{F}^1_X \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X \arrow[rr, "\gamma"] & & G(n+1,V) \end{tikzcd}$$

where $\pi(e_0,\dots,e_{n+a}) = [e_0]$ and $\rho(e_0,\dots,e_{n+a}) = [e_0 \wedge \dots \wedge e_n]$. Here the geometric object of interest is the incidence correspondence $\Gamma(X)$ defined in (3.2), but we work on $\mathcal{F}^1$ in order to use the Maurer-Cartan forms and Maurer-Cartan equation.

If $f = (e_0,\dots,e_{n+a}) \in \mathcal{F}^1$ is a general point, then $\dim \gamma(X) = \mathrm{rank}\, \rho_{*f}$ and moreover $\pi_*(\ker \rho_{*f}) = T_xF$.

Let $E = \{e_0, \ldots, e_n\}$. We calculate
$$\rho_{*f} = \omega^\mu_\alpha \otimes e^\alpha \otimes (e_\mu \bmod E).$$
Recall that the ω^μ_α are semi-basic for π; thus assertion 1 and the "conversely" part of the proposition follow, as Exercise 3.2.12.2 implies $T_x F = \operatorname{Singloc} |II_{X,x}|$.

We now implement a key idea in the method of the moving frame: now that we have more geometric information, we adapt frames further to reflect this information.

Let $\mathcal{F}^\gamma \subset \mathcal{F}^1$ be the subbundle of the first-order adapted frame bundle consisting of frames preserving the flag
$$\hat{x} \subset \{\hat{x}, \operatorname{Singloc} |II|_x\} \subset \hat{T} \subset V.$$
Let $\{\underline{e}_1, \ldots, \underline{e}_f\} = \operatorname{Singloc} |II|$. Use additional index ranges $1 \leq s, t \leq f$, $f + 1 \leq i, j, k \leq n$. In indices, $\mathcal{F}^\gamma$ is the set of frames preserving the flag:
$$\{e_0\} \subset \{e_0, e_s\} \subset \{e_0, e_s, e_j\} \subset \{e_0, e_s, e_j, e_\mu\}.$$
Exercise 3.4.3: Show that our adaptations have the effect that the pull-backs of the forms ω^μ_s to $\mathcal{F}^\gamma$ are zero and that $\operatorname{Singloc} |II| = \{\omega^j_0\}^\perp = \{\underline{e}_s\}$. (Recall that $S^*(\omega^\alpha_0)(\underline{e}_\beta) = \delta^\alpha_\beta$ for any section $S : X \to \mathcal{F}^1$.)

We will show that the fibers are indeed linear spaces.

Using the Maurer-Cartan equation, we see that
$$d\omega^j_0 \equiv -\omega^j_s \wedge \omega^s_0 \bmod \{\omega^i_0\}.$$
We claim that $\omega^j_s \equiv 0 \bmod \{\omega^i_0\}$ for all j, s. This is equivalent to asserting that the distribution $\operatorname{Singloc} |II|$ is integrable, which we already know by the argument above, but derive directly as follows:

Differentiating $\omega^\mu_s = 0$, we obtain
$$0 = d\omega^\mu_s = -\omega^\mu_j \wedge \omega^j_s. \tag{3.4}$$

Since $\omega^\mu_j = q^\mu_{jk} \omega^k_0$, applying the Cartan Lemma to (3.4) shows that, for each j, s, we have
$$\omega^j_s \equiv 0 \bmod \{\omega^i_0\}.$$

By Exercise 3.2.12.2, to show that the integral manifolds of the distribution (which we know to be the fibers F by our first argument) are linear spaces, it will suffice to show that $II_{F,[e_0]} = 0$. Because $\omega^\mu_s = 0$ when pulled back to $\mathcal{F}^\gamma$ and we may restrict $\mathcal{F}^\gamma$ to be a frame bundle over F, we have
$$II_{F,[e_0]} = \omega^j_s \omega^s_0 \otimes \underline{e}_j.$$
But $\omega^j_s \equiv 0 \bmod \{\omega^k_0\}$ and the ω^k_0 pulled back to F are zero; hence $II_F = 0$, and the proposition is proved. $\square$

Proof of Theorem 3.4.2. Let $p = [u^0 e_0 + u^s e_s] \in F$. We calculate $\tilde{T}_p X$. Since $F = \mathbb{P}\{e_0, e_s\} \subset \tilde{T}_p X$ for all $p \in F$, we can work modulo $\{e_0, e_s\}$,

$$\begin{aligned} dp &\equiv (u^0 \omega_0^j + u^s \omega_s^j) e_j \mod\{e_0, e_s\} \\ &\equiv (u^0 \delta_j^k + u^s C_{sj}^k) \omega_0^j e_k \mod\{e_0, e_s\}. \end{aligned}$$

Thus, p is a smooth point of X iff the matrix $(u^0 \delta_j^k + u^s C_{sj}^k)$ is invertible. Moreover, we see that in this case, $\tilde{T}_p X = \tilde{T}_{[e_0]} X$.

Let $\mathrm{Mat}_{m \times n}(\mathbb{C})$ denote the set of $m \times n$ complex matrices. Consider the linear map $i : \hat{F} \to \mathrm{Mat}_{(n-f)\times(n-f)}(\mathbb{C})$ given by $(u^0, u^s) \mapsto (u^0 \delta_j^k + u^s C_{sj}^k)$. Consider det, the determinant on $\mathrm{Mat}_{(n-f)\times(n-f)}(\mathbb{C})$, which is a polynomial of degree $n-f$. Note that $i^*(\det)$ is not identically zero, because $i(1, 0, \dots, 0)$ is the identity matrix. Thus, its zero set is a codimension one subset of F, and this zero set is the focal hypersurface. □

We can refine our observation about the focal hypersurface as follows: The matrix $i(u^0, u^s)$ could be zero for some (u^0, u^s). This possibility will occur if, considering each $C_s = C_{sk}^j$ as a matrix, the C_s's fail to be linearly independent. Consider new indices $1 \leq \rho \leq f'$, $f'+1 \leq \xi \leq f$ such that the C_ρ's are linearly independent and the C_ξ's are zero. Then the focal hypersurface is a cone with vertex the linear subspace $\mathbb{P}\{e_\xi\}$.

Exercises 3.4.4:

1. Show that tangential varieties have degenerate Gauss maps. ◎
2. Show that joins have degenerate Gauss maps. ◎
3. Let $M_r \subset \mathbb{P}(M_{m \times n})$ denote the projectivization of the set of matrices of rank at most r. What is the rank of the Gauss map of M_r? ◎

Remark 3.4.5. Variations of the above constructions produce additional examples of varieties with degenerate Gauss mappings, but not all of them. It is an open, vaguely posed, problem to classify varieties of dimension greater than two with degenerate Gauss mappings. Recent progress on this question has been made by Akivis and Goldberg [**4**], and by Piontkowski [**130**]. We discuss varieties with degenerate Gauss mappings more in §3.12. For now we consider the case of surfaces.

Classification of surfaces with degenerate Gauss mappings.

Theorem 3.4.6. *Let $X^2 \subset \mathbb{P}V$ be a variety with degenerate Gauss mapping. Then X is one of the following:*

i. a linearly embedded $\mathbb{P}^2$;

ii. a cone over a curve;

iii. the tangential variety to a curve.

This theorem is due to C. Segre ([**137**], p. 105). Note that (i) is a special case of both (ii) and (iii). We will see in the proof that it is the only intersection of those cases.

Proof. Assume that $X \neq \mathbb{P}^2$, so $II \neq 0$. Let $M \subset X_{\mathsf{smooth}}$ be an open subset where γ is of constant rank one. Let $\tilde{\mathcal{F}} \to M$ be the subbundle of $\mathcal{F}^1$ adapted additionally such that $\underline{e}_1 = \operatorname{Singloc}|II|$ and $II(\underline{e}_2, \underline{e}_2) = \underline{e}_3$. We have

$$\omega_1^3 = 0,$$
$$\omega_2^3 = \omega_0^2.$$

Thus

$$0 = d\omega_1^3 = -\omega_2^3 \wedge \omega_1^2,$$

so

$$\omega_1^2 = \lambda \omega_0^2$$

for some function $\lambda : \tilde{\mathcal{F}} \to \mathbb{C}$. Note that on $\tilde{\mathcal{F}}$ we have the fiber motion

$$e_1 \mapsto e_1 + g_1^0 e_0$$

which sends

$$\omega_1^2 \mapsto \omega_1^2 + g_1^0 \omega_0^2.$$

Thus by choosing $g_1^0 = -\lambda$, we may normalize $\lambda = 0$. Let $\mathcal{F}' \subset \tilde{\mathcal{F}}$ be the subbundle where $\lambda \equiv 0$.

The geometric meaning of our adaptation is as follows: our adaptation to $\tilde{\mathcal{F}}$ was such that the fiber of the Gauss map through $[e_0]$ was $\mathbb{P}\{e_0, e_1\}$. In this case, by Theorem 3.4.2 there is a unique singular point (the zero set of a degree one polynomial) in a general fiber. We have adapted frames such that this singular point is $[e_1]$.

Now, as we move $[e_0]$ in X transversely to the fiber of the Gauss map there are two possibilities: First, $[e_1]$ could stay fixed, in which case X must be a cone over $[e_1]$, because then $[e_1] \in \tilde{T}_x M$ for all $x \in M$. Otherwise, $[e_1]$ sweeps out a one-dimensional variety, i.e., a curve in X. We will show that, in the second case, X is the tangential variety to this curve. On $\mathcal{F}'$ we have

$$0 = d\omega_1^2 = -\omega_0^2 \wedge \omega_1^0,$$

which implies

$$\omega_1^0 = a\omega_0^2$$

for some function $a : \mathcal{F}' \to \mathbb{C}$. Note that a cannot be normalized to zero. Consider, on $\mathcal{F}^1$,

$$de_1 = e_0 \omega_1^0 + e_1 \omega_1^1.$$

If $a \equiv 0$, then $de_1 \equiv 0 \bmod e_1$, so $[e_1] \in \mathbb{P}V$ is a fixed point independent of $x \in M$ and X is a cone over $[e_1]$, as described above.

If a is not identically zero, then $de_1 \equiv 0 \mod e_0, e_1$, so the points $[e_1]$ sweep out a curve C such that $[e_0] \in \tilde{T}_{[e_1]}C$. In this case X is the tangential variety of C, because $[e_0]$ is a general point of X and $\dim \tau(C) = 2$. $\square$

Noticing that in the proof of Theorem 3.4.6 we only used the Frobenius Theorem, with additional work one can deduce

Theorem 3.4.7. *Let $M^2 \subset \mathbb{E}^3$ be a surface with Gauss curvature $K \equiv 0$. Then M minus a closed subset is the union of open subsets of the following types of surfaces:*

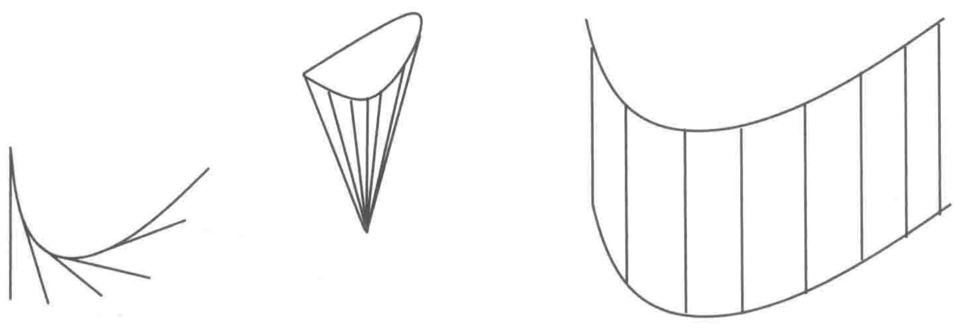
i. a (generalized) cylinder, i.e., a union of lines perpendicular to a plane curve;

ii. a (generalized) cone, i.e., the union of lines connecting a fixed point to a plane curve (minus the point);

iii. the union of tangent rays to a curve.

If M is complete, then it is a (generalized) cylinder.

See ([**142**], vol. III) for the proof.



A characterization of Gauss images. Recall that if $Y \subset G(k,V)$ is a subvariety, then for any $E \in Y$, $T_EY \subset E^* \otimes V/E$.

Theorem 3.4.8. *Let $Y \subset G(n+1,V)$ be an r-dimensional subvariety. Fix a general point $E \in Y$ and let $F(E) = \mathbb{P}(\bigcap_{v \in T_EY} \ker(v)) \subset \mathbb{P}E$. Then Y is a Gauss image, i.e., $Y = \gamma(X)$ for some $X^n \subset \mathbb{P}V$, iff*

i. $\dim F(E) = n - r$

and

ii. $\dim(\bigcup_{E \in Y} F(E)) = n$.

In this case $X = (\bigcup_{E \in Y} F(E))$.

This result was observed independently by Piontkowski and the second author around 1996.

Proof. It is clear that if $Y = \gamma(X)$ then (i) and (ii) are satisfied, as then $F(E)$ is just a general fiber of γ.

Now assume (i) and (ii) are satisfied and let $X = \bigcup_{E\in Y} F(E)$. Since unions and intersections of varieties are varieties, X is a variety. Let $1 \leq s,t,u \leq n-r$, $n-r+1 \leq j,k,l \leq n$, and adapt frames over each $x = [e_0] \in X_{\mathsf{general}}$ such that $F = \mathbb{P}\{e_0, e_s\}$, $E = \{e_0, e_s, e_j\}$.

The set of pairs $\{(F(E), \mathbb{P}E) \mid E \in Y\}$ gives a subvariety of the flag variety $\mathbb{F}_{\dim F(E),n}(V)$, and in particular, $dF(E) \subseteq E$ by Exercise 3.3.82. Expressed in frames, we have

$$de_0 \equiv 0 \operatorname{mod}\{e_0, e_s, e_j\}.$$

Thus $\hat{T}_x X \subseteq E$. Condition (ii) implies $\dim \hat{T}_x X = \dim E$, so we have equality. Since we are at a general point and X is a variety, we must have equality at all smooth points of X, and therefore $Y = \gamma(X)$. □

3.5. Higher-order differential invariants

In this section we determine a complete set of differential invariants for determining a submanifold of projective space $M \subset \mathbb{P}V$ up to $GL(V)$ equivalence. There are two types of such invariants. The first are the higher fundamental forms, which measure how M leaves its higher osculating spaces to first order. These differential invariants are defined as sections of natural vector bundles over M. The second type are relative differential invariants which we call *Fubini forms*, named after Fubini, who first used the cubic form F_3 to characterize quadric hypersurfaces (see Theorem 3.9.1). The Fubini forms are defined as sections of bundles over a frame bundle of M. For example, the Fubini cubic form, a third-order relative differential invariant, measures how M is infinitesimally leaving its embedded tangent space at a point to second order.

Throughout this section, $M \subset \mathbb{P}V$ is a complex submanifold of dimension n and $\mathcal{F}^1_M$ is its first-order adapted frame bundle as in §3.2. We continue the index ranges $1 \leq \alpha, \beta \leq n$ and $n+1 \leq \mu, \nu \leq n+a$.

The Fubini cubic form. Unlike with Euclidean geometry, the projective second fundamental form II is not enough information to determine local equivalence of complex submanifolds up to projective transformations (e.g., the case of hypersurfaces discussed in Remark 3.2.9). Thus we differentiate further to find higher-order differential invariants:

$$\begin{aligned} 0 &= d(\omega^\mu_\alpha - q^\mu_{\alpha\beta}\omega^\beta_0) \\ &= (-dq^\mu_{\alpha\beta} - q^\mu_{\alpha\beta}\omega^0_0 - q^\nu_{\alpha\beta}\omega^\mu_\nu + q^\mu_{\alpha\delta}\omega^\delta_\beta + q^\mu_{\beta\delta}\omega^\delta_\alpha) \wedge \omega^\beta_0, \end{aligned}$$

and using the Cartan Lemma, we conclude that there exist functions $r^\mu_{\alpha\beta\gamma}$, defined on $\mathcal{F}^1$, symmetric in their lower indices, and satisfying

$$r^\mu_{\alpha\beta\gamma}\omega^\gamma_0 = -dq^\mu_{\alpha\beta} - q^\mu_{\alpha\beta}\omega^0_0 - q^\nu_{\alpha\beta}\omega^\mu_\nu + q^\mu_{\alpha\delta}\omega^\delta_\beta + q^\mu_{\beta\delta}\omega^\delta_\alpha. \tag{3.5}$$

For $f \in \mathcal{F}^1$, define $F_3 = (F_3)_f \in \pi^*(S^3T^*M \otimes NM)$ to be

$$F_3 = r^\mu_{\alpha\beta\gamma}\omega^\alpha_0\omega^\beta_0\omega^\gamma_0 \otimes \underline{e}_\mu. \tag{3.6}$$

Moving in the fiber by the block diagonal matrices $g^0_0, g^\alpha_\beta, g^\mu_\nu$ does not change F_3. However, F_3 does *not* descend to be a well-defined section of $S^3T^*M \otimes NM$, because if $(\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\mu)$ is a new frame with

$$\begin{aligned} \tilde{e}_\mu &= e_\mu + g^0_\mu e_0 + g^\alpha_\mu e_\alpha, \\ \tilde{e}_\alpha &= e_\alpha + g^0_\alpha e_0, \end{aligned} \tag{3.7}$$

then

$$\tilde{r}^\mu_{\alpha\beta\gamma} = r^\mu_{\alpha\beta\gamma} + \mathfrak{S}_{\alpha\beta\gamma} g^0_\alpha q^\mu_{\beta\gamma} + \mathfrak{S}_{\alpha\beta\gamma} g^\delta_\nu q^\nu_{\alpha\beta} q^\mu_{\gamma\delta}, \tag{3.8}$$

where $\mathfrak{S}_{\alpha\beta\gamma}$ is cyclic summation in the fixed indices α, β, γ.

As mentioned above, $F_3 \in \Gamma(\mathcal{F}^1, \pi^*(S^3T^*M \otimes NM))$ is an example of a *relative differential invariant*. We will use the notation $\Delta r^\mu_{\alpha\beta\gamma}$ to denote the change in $r^\mu_{\alpha\beta\gamma}$ by a fiber motion of the type in (3.7). By (3.8),

$$\Delta r^\mu_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma}(g^0_\alpha q^\mu_{\beta\gamma} + g^\delta_\nu q^\nu_{\alpha\beta} q^\mu_{\gamma\delta}).$$

Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface and let $x \in X_{\mathsf{smooth}}$. Then $\mathrm{Baseloc}\,|II_{X,x}|$ defines a quadric hypersurface in $\mathbb{P}T_xX$. The space of cubic polynomials $|F_3|$ is not well-defined, but the ideal generated by $|II|, |F_3|$ is well-defined; thus there is a well-defined subvariety in $\mathbb{P}T_xX$ that is the intersection of a quadric hypersurface and a cubic hypersurface. We will see in §3.9 that if this intersection is simply the quadric hypersurface itself at a general point and the quadric has rank greater than one, then X itself must be a quadric hypersurface.

Remark 3.5.1. One can compare F_3 to the covariant derivative ∇II of the second fundamental form of a submanifold Y of a Riemannian manifold Z, which is a well-defined section of $S^3T^*Y \otimes NY$. In fact the forms $s^*(F_3)$, as $s : M \to \mathcal{F}^1$ varies over local sections, is the set of covariant derivatives of Riemannian metrics on $Z = \mathbb{P}V$ compatible with the linear structure.

Remark 3.5.2. If one uses local coordinates, then the third-order terms in a Taylor series expansion of X provide the coefficients of the cubic form. See §3.7 below.

The third fundamental form. If $II : S^2T_xM \to N_xM$ is not surjective, then one can refine the flag $\hat{x} \subset \hat{T}_xM \subset V$ to obtain a refined invariant from F_3. Let $\hat{T}^{(2)}_x M = \hat{T}_xM + \widehat{II}(S^2T_xM)$. Here we regard $\widehat{II}(S^2T_xM) \subset V/\hat{T}_xM$, so the sum of the two spaces determines a well-defined subspace of V.

Let $\mathcal{F}^2 \to M$ be the bundle of frames adapted to the flag

$$\hat{x} \subset \hat{T}_xM \subset \hat{T}^{(2)}_x M \subset V.$$

Use indices $n+1 \le \xi \le \dim \hat{T}^{(2)}_x M - 1$, $\dim \hat{T}^{(2)}_x M \le \phi \le n+a = \dim \mathbb{P}V$. Thus, frames $f \in \mathcal{F}^2$ are of the form $f = (e_0, e_\alpha, e_\xi, e_\phi)$ with $\hat{T}^{(2)} = \{e_0, e_\alpha, e_\xi\}$.

Exercises 3.5.3:
1. Show that $\omega^\phi_\alpha = 0$ on $\mathcal{F}^2$.
2. Show that the forms ω^ϕ_ξ are semi-basic when pulled back to $\mathcal{F}2$.
3. There exist functions $q^\phi_{\alpha\beta\gamma}$, symmetric in their lower indices, such that

$$\omega^\phi_\xi \omega^\xi_\alpha = q^\phi_{\alpha\beta\gamma}\omega^\beta_0\omega^\gamma_0. \tag{3.9}$$

Definition 3.5.4. Let $N_{2x}M = T_x\mathbb{P}V/(T_x{}^{(2)}M)$ and $\underline{e}_\phi = (e_\phi \bmod \hat{T}^{(2)}_x) \otimes e^0$. The tensor $\omega^\phi_\xi\omega^\xi_\alpha\omega^\alpha_0 \otimes \underline{e}_\phi \in \Gamma(\mathcal{F}^2, \pi^*(S^3T^*M \otimes N_2M)$ descends to a well-defined tensor $III : S^3T_xM \to N_{2x}M$, called the *projective third fundamental form.* The third fundamental form measures how M is moving away from its second osculating space to first order at x.

Exercises 3.5.5:
1. Show that the third fundamental form can also be defined at general points as the derivative of a second-order Gauss map

$$\gamma^{(2)} : M \to G(n + a_1 + 1, V),$$
$$x \mapsto \hat{T}^{(2)}_x.$$

Note that in general, $\gamma^{(2)}$ is just a rational map, i.e., only defined on a Zariski open subset of M.
2. Give an interpretation of III similar to the "quadratic part of the singularity of $X \cap H$ at x" interpretation of II. When is the cubic part of a singularity well-defined?

At general points, there is a severe restriction on the third fundamental form:

Theorem 3.5.6 (Cartan [**28**], p.377). *Let $M^n \subset \mathbb{P}V$ be an analytic submanifold. Then at general points*

$$III(N^*_{2,x}M) \subset S^3T^*_xM \cap (T^*_xM \otimes II(N^*_xM)). \tag{3.10}$$

We call (3.10) the *prolongation property.* It is ubiquitous in the theory of exterior differential systems.

Corollary 3.5.7. *If $X^n \subset \mathbb{P}V$ is a variety, $x \in X$ is a general point, $\dim |II_{X,x}| = 1$ and $\operatorname{rank} II(N_x^* X) > 1$, then X^n is a hypersurface in some $\mathbb{P}^{n+1}$.*

Exercises 3.5.8:

1. Prove Theorem 3.5.6. ⊚
2. Show that the prolongation property is equivalent to the quadratic form $v \lrcorner P$ being in $II(N_x^* M)$ for every $v \in T_x M$ and every $P \in III(N_{2,x}^* M)$.
3. Prove Corollary 3.5.7. ⊚

The k-th fundamental form $\mathbb{FF}^k$. We define $N_{(k-1),x}M$ and $\mathbb{FF}^k_{x,M} \in S^k T_x M \otimes N_{(k-1),x}M$ inductively. Assume $\mathbb{FF}^{k-1}$ has been defined. Let

$$\hat{T}_x^{(k)} M = \hat{T}_x^{(k-1)} X + \widehat{\mathbb{FF}^{k-1}}(S^{k-1} T_x M)$$

and let $N_{(k-1),x}M = T_x \mathbb{P}V / T_x^{(k-1)} M$.

Exercises 3.5.9:

1. Define $\mathbb{FF}^k$ by defining a higher-order Gauss map and taking its derivative.
2. Define $\mathbb{FF}^k$ by refining the flag and differentiating, as we did with $III = \mathbb{FF}^3$.
3. Show the two procedures above give the same differential invariant, called the *k-th fundamental form*, which measures to first order how M is leaving its $(k-1)$-st osculating space.

Notation. Let $|\mathbb{FF}^k|_{x,M} = \mathbb{FF}^k(N^*_{k,x} M) \subset S^k T_x^* M$.

Exercise 3.5.10: Show that $|\mathbb{FF}^k|_{M,x} \subseteq (T_x^* M \otimes |\mathbb{FF}^{k-1}|_{x,M}) \cap S^k T_x^* M$, i.e., that the prolongation property persists to higher fundamental forms.

An effective way to calculate fundamental forms. Here is a method to calculate fundamental forms (communicated to us by M. Green) that will be particularly useful. It generalizes the observation that if $de_0 \equiv \omega_0^\alpha e_\alpha \bmod e_0$ and $de_\alpha \equiv \omega_\alpha^\mu e_\mu \bmod \{e_0, e_\alpha\}$, then $II_{[e_0]} = \omega_\alpha^\mu \omega_0^\alpha \otimes \underline{e}_\mu$.

Define a series of maps

$$\underline{d}^k e_0 : (T\mathcal{F}^1)^{\otimes k} \to V / \operatorname{Image}(\underline{d}^0 e_0, \dots, \underline{d}^{k-1} e_0)$$

as follows: Let d denote exterior differentiation, let $\underline{d}^0 e_0 = e_0$ and let $\underline{d}^1 e_0 = de_0 \bmod e_0$. If $v_1, \dots, v_k \in T_f \mathcal{F}^1$, extend $v_1, \dots, v_k$ to holomorphic vector fields $\tilde{v}_1, \dots, \tilde{v}_k$ in some neighborhood of f. Let

$$\underline{d}^k e_0(v_1, \dots, v_k) := v_1 \lrcorner\, d(\tilde{v}_2 \lrcorner \dots d(\tilde{v}_k \lrcorner\, de_0) \bmod \pi_k{}^{-1}(\operatorname{Image} \underline{d}^{k-1} e_0) \tag{3.11}$$

where $\pi_k : V \to V/(\mathrm{Image}\{\underline{d}^0 e_0, \dots, \underline{d}^{k-1} e_0\})$ is the projection, and $\lrcorner$ denotes the contraction $T \times T^{*\otimes l} \to T^{*\otimes l-1}$. The proof that (3.11) is independent of the choice of extension to vector fields is standard (see, e.g., [**142**]). Then $\underline{d}^k e_0 \otimes e^0 = \mathbb{FF}^k$.

For example, keeping the index ranges on $\mathcal{F}^2$ that we used above to study the third fundamental form, we have

$$\begin{aligned}
\underline{d}^1 e_0 &\equiv \omega_0^\alpha e_\alpha \bmod\{e_0\},\\
\underline{d}^2 e_0 &= \omega_0^\alpha de_\alpha \bmod\{e_0, e_\alpha\}\\
&= \omega_0^\alpha \omega_\alpha^\xi e_\xi \bmod\{e_0, e_\alpha\},\\
\underline{d}^3 e_0 &= \omega_0^\alpha \omega_\alpha^\xi de_\xi \bmod\{e_0, e_\alpha, e_\xi\}\\
&= \omega_0^\alpha \omega_\alpha^\xi \omega_\xi^\phi e_\phi \bmod\{e_0, e_\alpha, e_\xi\}.
\end{aligned}$$

Algebraic definition of fundamental forms via spectral sequences. For those familiar with spectral sequences, here is yet another definition of the fundamental forms due to M. Green, [**65**]. This approach also works to define the relative differential invariants, see [**65**]. The maps $\underline{d}^k$ can be defined more algebraically as follows:

The quotient map

$$V^* \to V^*/\hat{x}^\perp = \mathcal{O}_{\mathbb{P}V}(1)_x$$

gives rise to a spectral sequence of a filtered complex by letting

$$\begin{aligned}
F^0K^0 &= V^*, & F^0K^1 &= \mathcal{O}_X(1)_x,\\
F^1K^0 &= 0, & F^pK^1 &= \mathfrak{m}_x^p(1),
\end{aligned}$$

where $\mathfrak{m}_x$ denotes the maximal ideal of functions vanishing at x, and $\mathfrak{m}_x^p$ denotes its p-th power (the functions vanishing to order $p-1$ at x). We let $F^p = F^pK^1$ for $p \geq 0$ and $T^* = T_x^*X$, etc. The maps are

$$\begin{aligned}
\underline{d}^0 &: V^* \to F^0/F^1 = \mathcal{O}_{X,x}(1)/\mathfrak{m}_x(1) \simeq \mathbb{C},\\
\underline{d}^1 &: \ker \underline{d}^0 \to F^1/F^2 = \mathfrak{m}_x(1)/\mathfrak{m}_x^2(1) \simeq T^*(1),\\
\underline{d}^2 &: \ker \underline{d}^1 \to F^2/F^3 = \mathfrak{m}_x^2(1)/\mathfrak{m}_x^3(1) \simeq (S^2T^*)(1),\\
&\vdots
\end{aligned}$$

3.6. Fundamental forms of some homogeneous varieties

Reduced frame bundles. Given $X^n \subset \mathbb{P}^{n+a}$, we may calculate our differential invariants by taking any section $s : X \to \mathcal{F}^1$ and pulling back the relevant forms. In particular, if there is a natural subbundle of $\mathcal{F}^1$ to work on,

we may restrict our frames to the subbundle without losing any geometric information.

In the case of homogeneous varieties $X = G/P \subset \mathbb{P}V$, we naturally have an embedding $G \to GL(V)$ which determines an identification $GL(V) \simeq \mathcal{F}_{\mathbb{P}V}$. With this identification $G \subset GL(V)$ becomes a subbundle of $\mathcal{F}_{\mathbb{P}V}$, and we will often work with this subbundle.

A second observation is that in practice we don't even need to explicitly calculate the embedding $G \to GL(V)$, because V is often a tensor space constructed from another representation of G. In the next two examples, V is respectively S^dW and Λ^kW, and thus natural bases for V are obtained from those of W. In particular, if $(e_0, \dots, e_n)$ is a basis of W, we apply our operator $\underline{d}$ to $e_0 \circ \cdots \circ e_0$ and $e_0 \wedge e_1 \wedge \cdots \wedge e_{k-1}$.

Fundamental forms of Veronese varieties. Let $W = \mathbb{C}^{n+1}$, let $V = S^pW$, and denote bases of W by (e_0, e_α), $1 \leq \alpha \leq n$, so the entries of the Maurer-Cartan form of $GL(W)$ are $\omega_0^0, \omega_\beta^0, \omega_0^\alpha, \omega_\beta^\alpha$. Let $x = [(e_0)^p]$. Let $\rho_p : GL(W) \to GL(S^pW)$ denote the natural inclusion. We work with the bundle of $\rho_p(GL(W)) \subset GL(S^pW)$ frames. Using the Leibniz rule $d(e_A \circ e_B) = de_A \circ e_B + e_A \circ de_B$, we have

$$\begin{aligned}
&\underline{d}e_0^p \equiv p\omega_0^\alpha e_\alpha e_0^{p-1},\\
&\underline{d}^2 e_0^p \equiv p(p-1)\omega_0^\alpha \omega_0^\beta e_\alpha e_\beta e_0^{p-2},\\
&\underline{d}^k e_0^p \equiv p(p-1)\dots(p-k+1)\omega_0^{\alpha_1}\dots\omega_0^{\alpha_k} e_{\alpha_1}\dots e_{\alpha_k} e_0^{p-k}.
\end{aligned}$$

Thus

$$\begin{aligned}
&|\mathbb{FF}^k_{v_p(\mathbb{P}W),[e_0^p]}| = \mathbb{P}S^k T^*_{[e_0^p]} v_p(\mathbb{P}W),\ k \leq p,\\
&\mathbb{FF}^k_{v_p(\mathbb{P}W)} = 0,\ k > p.
\end{aligned}$$

Fundamental forms of Grassmannians. Let $W = \mathbb{C}^n$ and let $V = \Lambda^kW$. Consider the inclusion $GL(W) \subset GL(V)$ with the induced action

$$g(v_1 \wedge \ldots \wedge v_k) = (gv_1) \wedge \ldots \wedge (gv_k).$$

Use index ranges $1 \leq i, j \leq k$ and $k+1 \leq s, t \leq n$.

Let $\vec{E} = e_1 \wedge \ldots \wedge e_k \in \widehat{G}(k, W)$, let $\vec{E}_s^j = e_1 \wedge \ldots \wedge e_{j-1} \wedge e_s \wedge e_{j+1} \wedge \ldots e_k$ where e_j has been replaced by e_s, let $\vec{E}_{st}^{ij}$ be $\vec{E}$ with e_s replacing e_i and e_t replacing e_j, and so on. Then the $\vec{E}_{s_1 \dots s_p}^{j_1 \dots j_p}$, with $j_1 < j_2 < \ldots < j_p$, $s_1 < s_2 < \ldots < s_p$ and $1 \leq p \leq k$, give a basis of Λ^kW.

Now, using the Maurer-Cartan forms for the $GL(W)$-frame bundle, we take derivatives:

$$d\vec{E} \equiv \omega_i^s E_s^i \operatorname{mod} \vec{E},$$

$$dE_s^i \equiv \sum_{i<j,s<t} \omega_j^t \vec{E}_{st}^{ij} + \sum_{i>j,s>t} \omega_j^t \vec{E}_{st}^{ij} - \sum_{i<j,s>t} \omega_j^t \vec{E}_{st}^{ij} - \sum_{i>j,s<t} \omega_j^t \vec{E}_{st}^{ij}$$
$$\operatorname{mod} \hat{T}_E G(k,W).$$

Thus $\hat{T}_E G(k,W) \operatorname{mod} \vec{E}$ is spanned by the vectors $\vec{E}, \vec{E}_s^i$ and we recover the interpretation $T_E G(k,W) \simeq E^* \otimes (W/E)$. Similarly, $II(S^2 T_E G(k,W)) \subset N_E G(k,W)$ is spanned by the vectors $\vec{E}_{st}^{ij} \operatorname{mod} \hat{T}_E G(k,W)$ and we may thus identify $II(S^2 T_E G(k,W))$ with $\Lambda^2 E^* \otimes \Lambda^2(W/E)$ as $GL(E) \times GL(W/E)$-modules.

It is natural (and correct!) to guess that

$$II : N_E^* G(k,W) \to S^2 T_E^* G(k,W)$$

is equivariant (see [**110**] for a proof and representation-theoretic interpretation of the fundamental forms of rational homogeneous varieties). Thus we expect that

$$\operatorname{Baseloc} |II_{G(k,W),E}| = \operatorname{Seg}(\mathbb{P}E^* \times \mathbb{P}(W/E)) \subset \mathbb{P}(E^* \otimes W/E),$$

because the Segre variety is the zero set of the 2×2 minors, i.e., of $\Lambda^2 E \otimes \Lambda^2(W/E)^* \subset S^2(E^* \otimes V/E)^*$.

We calculate explicitly that

$$II_{G(k,W),E} = \sum_{i<j,s<t} (\omega_i^s \omega_j^t - \omega_i^t \omega_j^s) \otimes \underline{E}_{st}^{ij},$$

proving that the quadrics in $|II|$ are the 2×2 minors of the matrices in $T_E G(k,W)$.

Exercise 3.6.1: Determine $\mathbb{FF}^l_{G(k,W)}$ and its base locus. ⊚

Since the Grassmannian is cut out by polynomials of degree two, the set $\operatorname{Baseloc} |II_{G(k,W),E}|$ consists of tangent directions to lines on $G(k,W)$ that pass through E. These lines may be seen explicitly. A point $[x \otimes y] \in \operatorname{Baseloc} |II_{G(k,W),E}| \subset \mathbb{P}(T_E G(k,W))$ determines a $(k-1)$-plane $F_x \subset E \subset V$ and a $(k+1)$-plane $L_y \subset V$ containing E, where $F_x = \ker x$ and $L_y = E + y$. The corresponding line in $G(k,W)$ is $\{E' \mid F_x \subset E' \subset L_y\}$.

Fundamental forms of Segre varieties. Let $U = \mathbb{C}^{a+1}, W = \mathbb{C}^{b+1}$, and consider the Segre embedding $\operatorname{Seg}(\mathbb{P}U \times \mathbb{P}W) \subset \mathbb{P}(U \otimes W)$. Here we use $G = GL(U) \times GL(W) \subset GL(U \otimes W)$ frames. Let $(e_0, \dots, e_a)$ and $(f_0, \dots, f_b)$ denote bases of U, W, and let the respective Maurer-Cartan forms be denoted (ω) and (η). Use index ranges $1 \le \alpha \le a$, $1 \le j \le b$. Let

$[e_0 \otimes f_0] \in \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W)$. Note that $e_0 \otimes f_0, e_\alpha \otimes f_0, e_0 \otimes f_j, e_\alpha \otimes f_j$ form a basis of $U \otimes W$. We compute

$$\underline{d}(e_0 \otimes f_0) \equiv \omega_0^\alpha e_\alpha \otimes f_0 + \eta_0^j e_0 \otimes f_j \bmod e_0 \otimes f_0, \tag{3.12}$$

$$\underline{d}^2(e_0 \otimes f_0) \equiv \omega_0^\alpha \eta_0^j e_\alpha \otimes f_j \bmod \hat{T}_{[e_0 \otimes f_0]} \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W). \tag{3.13}$$

Equation (3.12) shows that

$$\hat{T}_{[e_0 \otimes f_0]} \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W) = e_0 \otimes W + U \otimes f_0.$$

Let $U' = (U \otimes f_0 \bmod e_0 \otimes f_0) \otimes (e^0 \otimes f^0)$ and $W' = (e_0 \otimes W \bmod e_0 \otimes f_0) \otimes (e^0 \otimes f^0)$. We have

$$T_{[e_0 \otimes f_0]} \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W) = U' \oplus W'.$$

Equation (3.13) shows that

$$|II|_{\mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W),[e_0 \otimes f_0]} = U'^* \otimes W'^* \subset S^2 T^*_{[e_0 \otimes f_0]} \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W)$$

and that $III_{\mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W)}$ is zero. In particular,

$$\mathrm{Baseloc}\, |II|_{\mathrm{Seg}(\mathbb{P}U \times \mathbb{P}W),[e_0 \otimes f_0]} = \mathbb{P}U' \sqcup \mathbb{P}W'.$$

Note that here again, II is equivariant. We have $S^2 T^* = S^2(U' \oplus W')^* = S^2 U'^* \oplus (U'^* \otimes W'^*) \oplus S^2 W'^*$, and $II : N^* \to S^2 T^*$ is just the identity map on the middle factor.

Exercises 3.6.2:

1. Consider $X = \mathrm{Seg}(\mathbb{P}^{r_1} \times \ldots \times \mathbb{P}^{r_d}) = \mathrm{Seg}(\mathbb{P}W_1 \times \ldots \times \mathbb{P}W_d)$, $\dim W_i = r_i$. Use the $GL(W_1) \times \ldots \times GL(W_d) \subset GL(W_1 \otimes \cdots \otimes W_d)$ frame bundle in the following computations:
(a) Determine the tangent space and second fundamental form at a point of X. Interpret your results.
(b) Compute $\mathbb{FF}^k$ for all k. In particular, show that the last nonzero fundamental form is $\mathbb{FF}^d$.

2. Let W be equipped with a nondegenerate quadratic form Q. Consider $G_Q(k, W) \subset G(k, W) \subset \mathbb{P}(\Lambda^k W)$, the Grassmannian of k-planes isotropic for Q. Show that $T_E G_Q(k, W) \simeq E^* \otimes (E^\perp / E) \oplus \Lambda^2(V/E^\perp)$, and use this to calculate $\dim G_Q(k, W)$.

3. Let W be equipped with a nondegenerate symplectic form ω. Consider $G_\omega(k, W)$, and calculate $T_E G_\omega(k, W)$ and $\dim G_\omega(k, W)$. ⊚

Fundamental forms of Segre products of varieties.

Proposition 3.6.3. *Let $X_j \subset \mathbb{P}W_j$, $1 \le j \le r$, be varieties, let $x_j \in X_j$, and let $y = [x_1 \otimes \ldots \otimes x_r] \in Y := \mathrm{Seg}(X_1 \times \ldots \times X_r) \subset \mathbb{P}(W_1 \otimes \cdots \otimes W_r)$. Then*

$$\hat{T}_y Y = \Sigma_j x_1 \otimes \ldots \otimes x_{j-1} \otimes \hat{T}_{x_j} X_j \otimes x_{j+1} \otimes \ldots \otimes x_r,$$

$$\begin{aligned} II_{Y,y} = {} & \Sigma_{j<k} x_1 \otimes \ldots \otimes x_{j-1} \otimes T^*_{x_j} X_j \otimes x_{j+1} \ldots \otimes x_{k-1} \otimes T^*_{x_k} X_k \otimes x_{k+1} \otimes \ldots \otimes x_r \\ & + \Sigma_j x_1 \otimes \ldots \otimes x_{j-1} \otimes II_{X_j, x_j} \otimes x_{j+1} \otimes \ldots \otimes x_r. \end{aligned}$$

Exercise 3.6.4: Prove the proposition.

Exercise 3.6.5: Let $M_r \subset \mathbb{P}(M_{m\times n})$ denote the (projectivization of the) space of $m \times n$ matrices of rank at most r. Calculate $\hat{T}_x M_r$, where x has rank exactly r. Verify that the smooth points of M_r consist of the matrices of rank exactly r. ◎

The Severi varieties. This subsection has less detail than others. It will be used in the proof of Zak's theorem on Severi varieties and may be skipped on a first reading.

Let $\mathbb{A}$ denote the complexification $\mathbb{A}_{\mathbb{R}} \otimes \mathbb{C}$ of a real division algebra $\mathbb{A}_{\mathbb{R}}$ (see §A.5), and let $\mathcal{H}_{\mathbb{R}}$ denote the $\mathbb{A}_{\mathbb{R}}$-Hermitian forms on $\mathbb{A}^3_{\mathbb{R}}$, i.e., the 3×3 $\mathbb{A}_{\mathbb{R}}$-Hermitian matrices. If $x \in \mathcal{H}_{\mathbb{R}}$, then we may write

$$x = \begin{pmatrix} r_1 & \bar{u}_1 & \bar{u}_2 \\ u_1 & r_2 & \bar{u}_3 \\ u_2 & u_3 & r_3 \end{pmatrix}, \qquad r_i \in \mathbb{R},\ u_i \in \mathbb{A}_{\mathbb{R}}.$$

Let $\mathcal{H} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$.

Exercise 3.6.6: Verify that the notion of x^2 and x^3 make sense. In the case of the octonions, one needs to use the Moufang identities (A.12).

Define a cubic form det on $\mathcal{H}$ by

$$\mathsf{det}(x) := \frac{1}{6}\left((\mathrm{trace}(x))^3 + 2\,\mathrm{trace}(x^3) - 3\,\mathrm{trace}(x)\,\mathrm{trace}(x^2)\right).$$

This det is just the usual determinant of a 3×3 matrix when $\mathbb{A} = \mathbb{C}$.

Now, considering $\mathcal{H}$ as a vector space over $\mathbb{C}$, let G be the subgroup of $GL(\mathcal{H}, \mathbb{C})$ preserving det, i.e., define

$$G := \{g \in GL(\mathcal{H}, \mathbb{C}) \mid \mathsf{det}(gx) = \mathsf{det}(x)\ \forall x \in \mathcal{H}\}.$$

For the four division algebras, the respective groups are:

$\mathbb{A}_{\mathbb{R}}$	G
$\mathbb{R}$	$SL(3,\mathbb{R})_{\mathbb{C}} = SL(3,\mathbb{C})$
$\mathbb{C}$	$SL(3,\mathbb{C})_{\mathbb{C}} = SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$
$\mathbb{H}$	$SL(3,\mathbb{H})_{\mathbb{C}} = SL(6,\mathbb{C})$
$\mathbb{O}$	'$SL(3,\mathbb{O})_{\mathbb{C}}$' $= E_6$

We take the above as the definition of E_6, and have written '$SL(3,\mathbb{O})_{\mathbb{C}}$' merely to be suggestive.

Definition 3.6.7. The group F_4 is the subgroup of E_6 preserving the quadratic form $Q(x,x) = \mathrm{trace}(x^2)$, where x^2 is the usual matrix multiplication and one must again use the Moufang identities (A.12) to be sure that Q is well-defined.

Exercise 3.6.8: Show that the action of F_4 on $\mathcal{H}$ preserves the line $\{\mathrm{Id}\}_{\mathbb{C}}$ and $\mathcal{H}_0 := \{x \in \mathcal{H} \mid \mathrm{trace}(x) = 0\}$. (In fact, F_4 acts irreducibly on both factors.)

The cubic form det tells us which elements of $\mathcal{H}$ are of less than full rank. One can also unambiguously define a notion of being rank one, either by taking 2×2 minors or by noting that under the G action each $x \in \mathcal{H}$ is diagonalizable and one can take as the rank of x the number of nonzero elements in the diagonalization of x. Let

$$X := \mathbb{P}\{\text{rank one elements of } \mathcal{H}\} = \mathbb{P}\{G\text{-orbit of any rank one matrix}\}.$$

Then $X = (\mathbb{A}_{\mathbb{R}}\mathbb{P}^2)_{\mathbb{C}}$, that is, the complexification of the space of $\mathbb{A}_{\mathbb{R}}$-lines in $\mathbb{A}^3_{\mathbb{R}}$. The four varieties $X \subset \mathbb{P}\mathcal{H}$ are called the *Severi varieties.*

Exercises 3.6.9:

1. Show that the first three Severi varieties are $v_2(\mathbb{P}^2) \subset \mathbb{P}^5, \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$, and $G(2,6) \subset \mathbb{P}^{14}$.
2. What are the analogous groups and varieties if one takes instead the 2×2 $\mathbb{A}$-Hermitian forms?
3. Let $\mathbb{O}\mathbb{P}^2_0 = \mathcal{H}_0 \cap \mathbb{O}\mathbb{P}^2$, where $\mathcal{H}_0 \subset \mathcal{H}$ is the set of traceless elements. Show that
$$\mathbb{O}\mathbb{P}^2_0 = \mathbb{P}\{x \in \mathcal{H}_0 \mid x^2 = 0\}$$
and deduce that it is a homogeneous space of the group F_4.
4. For each of the Severi varieties, calculate $\dim \sigma(X)$.

Fundamental forms of the Severi varieties. Here it is easier to use local coordinates. Choose affine coordinates based at $[p]$, where

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and denote the affine coordinates $u_1, u_2, u_3 \in \mathbb{A}, r_2, r_3 \in \mathbb{C}$, where the tangent space to p is $\{u_1, u_2\}$ (the span is taken over $\mathbb{C}$). In these coordinates:

$$r_2(u_1, u_2) = u_1\bar{u_1} \quad \text{because} \quad \det \begin{pmatrix} 1 & \bar{u_1} \\ u_1 & r_2 \end{pmatrix} = 0,$$

$$r_3(u_1, u_2) = u_2\bar{u_2} \quad \text{because} \quad \det \begin{pmatrix} 1 & \bar{u_2} \\ u_2 & r_3 \end{pmatrix} = 0,$$

$$u_3(u_1, u_2) = \bar{u_2}u_1 \quad \text{because} \quad \det \begin{pmatrix} 1 & \bar{u_1} \\ u_2 & u_3 \end{pmatrix} = 0,$$

where the last equation gives us one, two, four or eight quadratic forms. The determinants come from the vanishing of 2×2 minors that must be zero

to make the Severi variety consist only of rank one elements. In division algebra notation the second fundamental forms are

$$|II| = \mathbb{P}\{u_1\bar{u}_1, u_2\bar{u}_2, \bar{u}_2 u_1\}.$$

The spinor variety $\mathbb{S}_5$ *and its cousins.* An important series of homogeneous varieties is the *spinor varieties* described at the end of this section. The first spinor variety that does not coincide with a homogeneous variety that we have already discussed is $\mathbb{S}_5^{10} \subset \mathbb{P}^{15}$. It admits a description using the octonions (see §A.5) as follows: Let $\mathbb{C}^{16} = \mathbb{O}^2 \otimes \mathbb{C}$ have (complexified) octonionic coordinates u, v. Then $\mathbb{S}_5$ is defined by the equations $u\bar{u} = 0, v\bar{v} = 0$ and $u\bar{v} = 0$. Note that the last expression gives eight equations for the components of u and v.

Exercise 3.6.10: What are the varieties analogous to $\mathbb{S}_5$ for the other division algebras?

Minuscule varieties.

Definition 3.6.11. Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety where G is a complex semi-simple Lie group acting linearly on V and P is the subgroup stabilizing a point of X (such a P is called a *parabolic* subgroup). We say X is a *generalized minuscule variety* if T_xX contains no proper irreducible P-submodule. X is a *minuscule variety* if moreover G is simple and the embedding is minimal, which is equivalent to the embedding not being a Veronese re-embedding.

Alternatively, the generalized minuscule varieties are those admitting an Hermitian symmetric metric induced from a Fubini-Study metric on the ambient projective space, and the minuscule varieties are those for which the metric is irreducible and the embedding minimal.

Examples of minuscule varieties include the quadric hypersurfaces $Q^n \subset \mathbb{P}^{n+1}$, the Grassmannians $G(k, V)$, the spinor varieties $\mathbb{S}_m = Spin(2m)/P$ defined below, the Cayley plane $\mathbb{OP}^2$, and the Lagrangian Grassmannians $G_{\omega-\mathsf{null}}(m, 2m)$. In fact these are all the minuscule varieties except for $G_w(\mathbb{O}^3, \mathbb{O}^6)$, which is a homogeneous variety of E_7 (see **[112]** for a description of $G_w(\mathbb{O}^3, \mathbb{O}^6)$).

If we know the structure of the tangent space at a point of a minuscule variety as a P-module, the fundamental forms are easy to determine.

Proposition 3.6.12 (**[110]**). *Let $X = G/P \subset \mathbb{P}V$ be a minuscule variety and let $x \in X$. Let $H \subset P$ be a maximal semi-simple subgroup and consider T_xX as an H-module. Let $Y \subset \mathbb{P}T_xX$ be the closed H-orbit. Then* $\mathrm{Baseloc}\,|II_{X,x}| = Y$*, and moreover $|II_{X,x}| = I_2(Y)$.*

For example, once we know $T_E G(k, W) \simeq E^* \otimes W/E$ and $H = SL(E) \times SL(W/E)$, we may use this proposition to immediately conclude that

$$\text{Baseloc}\,|II_{G(k,W),E}| = \text{Seg}(\mathbb{P}E \times \mathbb{P}(W/E))$$

and the second fundamental form is spanned by the 2×2 minors.

Remark 3.6.13. For readers familiar with Dynkin diagrams, H and the H-module $T_x X$ can be determined pictorially: H is a semi-simple group with Lie algebra whose Dynkin diagram is that of G with the node corresponding to the root defining the maximal parabolic P removed. Moreover, if the diagram is simply laced, the H-module $T_x X$ is the irreducible module with highest weight the sum of the fundamental weights corresponding to the nodes of the Dynkin diagram adjacent to the deleted node.

For example, in the following picture, the parabolic defining $X = G(4,7)$ and its natural embedding corresponds to the darkened node in the Dynkin diagram of $A_7 = SL_8$, where to define the parabolic we omit the root corresponding to the node, and to define the embedding we take the representation with support the fundamental weight corresponding to the marked node. The $A_3 \times A_2 = SL_4 \times SL_3$ module $T_x X$ is pictured below it, and the same marked diagram gives $\text{Seg}(\mathbb{P}^3 \times \mathbb{P}^1) = \text{Baseloc}\,|II_{G(3,7)}|$.

Figure 1. Parabolic defining $X = G(4,7)$

Figure 2. $SL_4 \times SL_3$ module $T_x X$

Exercise 3.6.1 shows that equality in (3.10) holds for the examples of Grassmannians. In fact, equality holds for all minuscule varieties:

Theorem 3.6.14 ([**110**]). *Let $X = G/P \subset \mathbb{P}V$ be a minuscule variety and let $x \in X$. Then for $k \geq 2$,*

$$|\mathbb{FF}^{k+1}_{X,x}| = |\mathbb{FF}^2_{X,x}|^{(k-1)}.$$

This strict prolongation property fails for a general homogeneous variety, e.g., it does not hold for $G_{Q-\mathsf{null}}(k, m)$ for $2 \leq k < \lfloor \frac{m}{2} \rfloor$. This property has the following geometric consequence:

Corollary 3.6.15 ([**110**]). *Let X be a minuscule variety, and let $x \in X$. Then*

$$|\mathbb{FF}^k_{X,x}| = I_k(\sigma_{k-1}(\text{Baseloc}\,|\mathbb{FF}^2_{X,x}|)).$$

See [**110**] for proofs.

Spinor varieties and the spin representation. For $\mathfrak{sl}(W)$, the fundamental representations are just the exterior powers of the standard representation (see §A.6). For $\mathfrak{so}(W)$, assuming for simplicity that $\dim W = 2m$, fundamental representations are furnished by the exterior powers of the standard representation, up to $\Lambda^{m-2}W$. There are two other fundamental representations of $\mathfrak{so}(W)$, dual to one another, one of which we now describe. These representations are called the *spin representations*. In terms of Lie groups, the spin representations are representations of the simply-connected double cover of $SO(W)$, called $\mathrm{Spin}(W)$, but not of $SO(W)$.

We will use Corollary 3.6.15 to construct a model for the spin representation. Readers familiar with Dynkin diagrams may skip the first step, where the tangent space is calculated, thanks to 3.6.13.

Before starting, we examine the Grassmannian $G(k, W)$ from the perspective we will use to construct the spin representation. Say we knew Corollary 3.6.15 but were unaware of the Plücker embedding. We would then construct $V = \Lambda^k W$ as follows: fix $E \in G(k, W)$; then V is the sum of the successive quotient spaces $\Lambda^k E$ (which we identify with the point E), $E^* \otimes W/E = \Lambda^{k-1}E \otimes W/E$, $\Lambda^2 E^* \otimes \Lambda^2(W/E) = \Lambda^{k-2}E \otimes \Lambda^2(W/E)$,..., $\Lambda^k E^* \otimes \Lambda^k(W/E) = \Lambda^0 E \otimes \Lambda^k(W/E)$. Note that only the first space is well-defined as a subspace of $\Lambda^k W$, but the sum of the first p spaces is a well-defined subspace of $\Lambda^k W$ for all p, so we obtain a $\Lambda^k W$ equipped with a filtration.

We now apply the same method to recover the spin representation. The variety $G_{Q-\mathsf{null}}(m, \mathbb{C}^{2m})$ has two isomorphic components; we define $\mathbb{S}_m$ to be one of them.

Choose a basis of $V = \mathbb{C}^{2m}$ so that $Q = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. With respect to this basis $SO(V, Q)$ has Maurer-Cartan form

$$\begin{pmatrix} \omega^i_j & \omega^i_{m+k} \\ \omega^{n+l}_j & \omega^{m+k}_{m+l} \end{pmatrix},$$

where the index range is $1 \le i, j, k \le m$, and $\omega^{m+k}_{m+l} = -\omega^l_k, \omega^i_{n+k} = -\omega^k_{m+i}$ and $\omega^{m+l}_j = -\omega^{m+j}_l$.

Let $\vec{E} = e_1 \wedge \ldots \wedge e_m \in \hat{\mathbb{S}}_m$. By Remark 3.6.13 we could conclude that $T_E\mathbb{S}_m \simeq \Lambda^2 E^*$, or, computing as for the Grassmannians,

$$d\vec{E} \equiv \omega^{n+i}_j E^j_{n+i} \bmod E.$$

This determines $T_E\mathbb{S}$ as a linear subspace of $E^* \otimes V/E$. Since Q allows us to identify V/E with E^*, we may consider $T_E\mathbb{S}$ as a subspace of $E^* \otimes E^*$. The subspace is $\Lambda^2 E^*$ because the only relations among the forms ω^{n+i}_j are $\omega^{n+i}_j = -\omega^{n+j}_i$. Thus, $T_E\mathbb{S}_m = \Lambda^2 E^*$, and we note that $\dim \mathbb{S}_m = \binom{m}{2}$.

At this point we may apply Proposition 3.6.12, but even without the proposition we could calculate the second fundamental form of $\mathbb{S}$ using our frame bundle:

$$\underline{d}^2\vec{E} \equiv \omega_j^{n+i}\omega_l^{n+k}\vec{E}^{jl}_{n+i,n+k} \operatorname{mod} \hat{T}_E\mathbb{S}.$$

Thus

$$|II_{\mathbb{S},E}| \simeq \Lambda^4 E^* = I_2(G(2,E)).$$

Exercises 3.6.16:

1. Verify that $\operatorname{Baseloc}|II_{\mathbb{S}_m,E}| = G(2,E)$ by showing that $II(v,v) = 0$ if and only if $v = w_1 \wedge w_2$. ⊚
2. Calculate the higher fundamental forms of $\mathbb{S}_m$ via moving frames to verify that $|\mathbb{FF}^k| = \Lambda^{2k}E^* = I_k(\sigma_{k-1}(G(2,E)))$.

Adding up the images of the $\mathbb{FF}^k$'s, we see that $\mathbb{S}_m \subset \mathbb{P}(\Lambda^{even}E^*)$. We conclude that $\operatorname{Spin}(2m)$ and $\mathfrak{so}_{2m}$ act on $\Lambda^{even}\mathbb{C}^m$; either of these actions is called the *spin representation* or sometimes the *half-spin representation*.

In addition to the special construction for $\mathfrak{so}_{10}$ of the previous section, another model for the spin representation, using Clifford algebras, is described in Appendix A.

Exercise 3.6.17: Show that in the case of $\mathbb{S}_5$, our construction agrees with that in 3.6.

3.7. Higher-order Fubini forms

In this section we define the higher-order relative differential invariants which we call the *Fubini forms* and give some applications. We explain their use in the study of osculating hypersurfaces to a variety. We prove the classical Bertini Theorem for systems of quadrics and higher-order generalizations useful for projective geometry. More extensive applications are given in §3.8, §3.9 and §3.16.

Definition of the Fubini forms F_k. Let $M^n \subset \mathbb{P}^{n+a}$ be a submanifold, let $\pi : \mathcal{F}^1 \to M$ be its first-order adapted frame bundle, and continue the index ranges $1 \le \alpha, \beta \le n$ and $n+1 \le \mu, \nu \le n+a$.

Differentiating F_3, one obtains a fourth-order invariant

$$F_4 = r^\mu_{\alpha\beta\gamma\delta}\omega_0^\alpha\omega_0^\beta\omega_0^\gamma\omega_0^\delta \otimes \underline{e}_\mu \in \Gamma(\mathcal{F}^1, \pi^*(S^4T^*M \otimes NM))$$

whose coefficients are defined by

$$r^\mu_{\alpha\beta\gamma\delta}\omega_0^\delta = -dr^\mu_{\alpha\beta\gamma} - 2r^\mu_{\alpha\beta\gamma}\omega_0^0 - r^\nu_{\alpha\beta\gamma}\omega^\mu_\nu + \mathfrak{S}_{\alpha\beta\gamma}(r^\mu_{\alpha\beta\epsilon}\omega^\epsilon_\gamma + q^\mu_{\alpha\beta}\omega^0_\gamma - q^\mu_{\alpha\epsilon}q^\nu_{\beta\gamma}\omega^\epsilon_\nu).$$

The geometric interpretation of F_4 is that it measures how X leaves its embedded tangent space to third order.

As with F_3, motion in the fiber by a block diagonal matrix will not effect F_4. Under the fiber motion (3.7), the coefficients of F_4 vary as follows:

$$\tilde{r}^{\mu}_{\alpha\beta\gamma\delta} = r^{\mu}_{\alpha\beta\gamma\delta} + \mathfrak{S}_{\alpha\beta\gamma\delta}[g^0_{\alpha} r^{\mu}_{\beta\gamma\delta} + g^{\epsilon}_{\nu}(r^{\nu}_{\alpha\beta\gamma} q^{\mu}_{\delta\epsilon} + q^{\nu}_{\alpha\beta} r^{\mu}_{\gamma\delta\epsilon}) + g^0_{\nu} q^{\mu}_{\alpha\beta} q^{\nu}_{\gamma\delta}].$$

One continues, defining forms $F_k \in \Gamma(\mathcal{F}^1, \pi^*(S^kT^* \otimes N))$ for all k. The coefficients of F_l are defined by

$$\begin{aligned} r^{\mu}_{\alpha_1,\dots,\alpha_l}\omega^{\alpha_l}_0 = &- dr^{\mu}_{\alpha_1,\dots,\alpha_{l-1}} - (l-2) r^{\mu}_{\alpha_1,\dots,\alpha_{l-1}}\omega^0_0 - r^{\nu}_{\alpha_1,\dots,\alpha_{l-1}}\omega^{\mu}_{\nu} \\ &+ \mathfrak{S}_{\alpha_1,\dots,\alpha_{l-1}}[r^{\mu}_{\alpha_1,\dots,\alpha_{l-2}\beta}\omega^{\beta}_{\alpha_{l-1}} + (l-3) r^{\mu}_{\alpha_1,\dots,\alpha_{l-2}}\omega^0_{\alpha_{l-1}} \\ &- \sum_{p=1}^{l-3} r^{\mu}_{\alpha_1,\dots,\alpha_p,\beta} r^{\nu}_{\alpha_{p+1},\dots,\alpha_{l-1}}\omega^{\beta}_{\nu} \\ &- \sum_{p=2}^{l-3}(p-2+l) r^{\mu}_{\alpha_1,\dots,\alpha_p} r^{\nu}_{\alpha_{p+1},\dots,\alpha_{l-1}}\omega^0_{\nu}]. \end{aligned}$$

This formula corrects the typographical errors in (2.20) of [**102**] (where the proof is given without typographical errors).

Projective differential invariants in coordinates. Let $M \subset \mathbb{P}V$ be a submanifold and let $x^1, \dots, x^{n+a}$ be local coordinates such that locally M is given as a graph $x^{\mu} = f^{\mu}(x^1, \dots, x^n)$. Write $f^{\mu}_{x^{\alpha}} = \frac{\partial f^{\mu}}{\partial x^{\alpha}}$.

Consider the section of $\mathcal{F}^1_M$ defined by $e_{\alpha} = \frac{\partial}{\partial x^{\alpha}} + f^{\mu}_{x^{\alpha}}\frac{\partial}{\partial x^{\mu}}$, $e_{\mu} = \frac{\partial}{\partial x^{\mu}}$. In this framing $\omega^{\alpha}_0 = dx^{\alpha}$, or, more precisely, $s^*(\omega^{\alpha}_0) = dx^{\alpha}$. We compute

$$de_{\alpha} \equiv f^{\mu}_{x^{\alpha}x^{\beta}} dx^{\beta} \otimes \frac{\partial}{\partial x^{\mu}}.$$

In terms of the Maurer-Cartan form, we have

$$de_{\alpha} \equiv \omega^{\mu}_{\alpha} e_{\mu} \operatorname{mod}\{e_0, e_{\beta}\},$$

so $\omega^{\mu}_{\alpha} = f^{\mu}_{x^{\alpha}x^{\beta}}\omega^{\beta}_0$. Thus

$$II = f^{\mu}_{x^{\alpha}x^{\beta}} dx^{\alpha} dx^{\beta} \otimes e_{\mu}.$$

With a little more work, one can adapt frames so that

$$F_k = f^{\mu}_{x^{\alpha_1},\dots,x^{\alpha_k}} dx^{\alpha_1} \dots dx^{\alpha_k} \otimes e_{\mu},$$

i.e., *the coefficients of F_k are just the terms appearing in the k-th term of the Taylor series expansion of the functions locally defining M as a graph over its tangent space.*

Exercise 3.7.1: Show that there exists a section $s : M \to \mathcal{F}^1$ such that $s^*(F_3) = f^{\mu}_{x^{\alpha}x^{\beta}x^{\gamma}} dx^{\alpha} dx^{\beta} dx^{\gamma} \otimes \frac{\partial}{\partial x^{\mu}}$.

Corollary 3.7.2. *A submanifold $M \subset \mathbb{P}V$ is uniquely determined up to projective equivalence by the infinite sequence of relative differential invariants F_k.*

Proof. They determine the Taylor series. □

Remark 3.7.3. A projective variety $X \subset \mathbb{P}V$ will be uniquely determined up to projective equivalence by a finite number of the F_k, but it is subtle to determine exactly how many F_k are needed, even if one has the degrees of the defining equations of X. The simplest case is described in the next section.

Osculating hypersurfaces and Veronese re-embeddings. Given a variety $X \subset \mathbb{P}V$, and a point $x \in X$, how could we determine if X is contained in a hyperplane H? An obvious necessary condition is that $x \in H$, and moreover that $\tilde{T}_x X \subseteq H$, i.e., that X does not infinitesimally move away from H at x to first order. The next thing to check is that X does not infinitesimally move away from H at x to second order, i.e., that $\vec{H} \in \ker II_{X,x}$, where $\vec{H} \in N_x^* X$ is a vector corresponding to H. (Recall that $\mathbb{P}N_x^* X$ may be identified with the space of hyperplanes containing $\tilde{T}_x X$.) Continuing, $X \subseteq H$ iff $\vec{H} \in \ker \mathbb{FF}^k_{X,x}$ for all k. If x is a general point and X is a hypersurface, then we have seen (Exercise 3.2.12.2) that in fact it is sufficient to check that $\vec{H} \in \ker II_{X,x}$ (i.e., to check that $II_{X,x} = 0$). If the codimension of X is greater than one but still small and X is not too singular, then Corollary 3.14.3 will imply that the same conclusion holds. In this subsection and §3.16 we address the corresponding question for higher degree hypersurfaces. Our approach will be to reduce to the hyperplane case by re-embedding X via a Veronese mapping.

We first determine, for each d and small k, *a priori* bounds on the dimensions hypersurfaces of degree d osculating to order k at a smooth point of a variety $X^n \subset \mathbb{P}^{n+a}$. Recall that for degree one, the dimension of hypersurfaces of degree one (i.e., hyperplanes) osculating to order one (i.e., containing $\tilde{T}_x X$) at a smooth point $x \in X$ is fixed (equal to $a-1$). The dimension of the space of hyperplanes osculating to order two depends on the geometry of X and the point x. We show that a similar phenomenon happens when $d > 1$. We begin with $d = 2$.

We use the notation $F_0 = \mathbb{FF}^0 = \hat{x} \otimes \hat{x}^*$ and $F_1 = \mathbb{FF}^1 = \mathrm{Id}_{T_x X}$.

Fundamental forms of $v_2(X)$ and osculating quadrics. Assume $X \subset \mathbb{P}V$ is such that $II : S^2 T_x X \to N_x X$ is surjective at general points $x \in X$. Write $x = [e_0]$, $v_2(x) = [e_0 \circ e_0]$, and use the Leibniz rule applied to $e_0 \circ e_0$

to compute

$$
\begin{aligned}
\mathbb{FF}^1_{v_2(X),v_2(x)} &= 2F_1F_0|_{\hat{x}^{2\perp}},\\
\mathbb{FF}^2_{v_2(X),v_2(x)} &= 2(F_2F_0+F_1F_1)|_{(\hat{x}\hat{T})^\perp},\\
\mathbb{FF}^3_{v_2(X),v_2(x)} &= 2(F_3F_0+3F_2F_1)|_{\ker \mathbb{FF}^2_{v_2(X)}},\\
\mathbb{FF}^4_{v_2(X),v_2(x)} &= 2(F_4F_0+4F_3F_1+3F_2F_2)|_{\ker \mathbb{FF}^3_{v_2(X)}},\\
\mathbb{FF}^5_{v_2(X),v_2(x)} &= 2(F_5F_0+5F_4F_1+10F_3F_2)|_{\ker \mathbb{FF}^4_{v_2(X)}},\\
&\ \vdots
\end{aligned}
\tag{3.14}
$$

Here $F_j = F_{j,X,x}$ are the relative differential invariants of $X \subset \mathbb{P}V$, and $F_jF_k = F_j \circ F_k$.

We make several observations. First, note that $|\mathbb{FF}^1_{v_2(X),v_2(x)}| \simeq T^*_xX$, $|\mathbb{FF}^2_{v_2(X),v_2(x)}| \simeq S^2T^*_xX$, and thus the dimensions of their kernels are respectively $\binom{n+a+1}{2} - n$ and $\binom{n+a+1}{2} - n - \binom{n+1}{2}$. This says:

Independent of the variety X and the smooth point $x \in X$, the dimensions of the spaces of quadrics osculating to orders one and two at x are fixed.

If we choose a splitting of our flag $\hat{x} \subset \hat{T} \subset V$ and slightly abuse notation, writing $V = x + T + N$, then

$$S^2V^* = S^2x^* + x^*\otimes T^* + x^*\otimes N^* + S^2T^* + T^*\otimes N^* + S^2N^*.$$

The quadrics corresponding to the S^2N^* factor are necessarily in the kernel of $\mathbb{FF}^3_{v_2(X),v_2(x)}$, and this observation is independent of our choices, so we see that:

Independent of the variety X and the smooth point $x \in X$, there is a lower bound on the dimension of the space of quadrics osculating to order three at x.

The dimension of $\ker \mathbb{FF}^3_{v_2(X),v_2(x)}$ depends on the geometry of $F_2 = II_{X,x}$. Recall from Appendix A that the kernel of the symmetrization map $S^2T^*\otimes T^* \to S^3T^*$ is denoted $S_{21}T^*$. An element of $F_2\otimes F_1$ will fail to contribute to $\mathbb{FF}^3_{v_2(X)}$ iff it lies in $S_{21}T^*$, so

The dimension of $\ker \mathbb{FF}^3_{v_2(X),v_2(x)}$ *is guaranteed to be as small as possible if* $(|II|\otimes T^*) \cap S_{21}T^* = 0$.

Note that if $(|II|\otimes T^*) \cap S_{21}T^* \neq 0$, then there exists a relation $Q_1 l^1 + \cdots + Q_p l^p = 0$ with $Q_j \in |II|$ and $l^j \in T^*$. Such a relation is called a *linear syzygy* among the quadrics in $|II|$.

Assuming $\ker \mathbb{FF}^3_{v_2(X),v_2(x)}$ is as small as possible, $\ker \mathbb{FF}^4_{v_2(X),v_2(x)}$ will be assured to be as small as possible if the map $|II|\otimes|II| \to S^4T^*$ has no kernel.

Exercise 3.7.4: Show that the map $|II|\otimes|II| \to S^4T^*$ has no kernel if there are no linear syzygies among the quadrics in $|II|$.

$v_d(X)$ and osculating hypersurfaces of degree d. We have the following generalization of (3.14):

Proposition 3.7.5 ([**102**]). *The fundamental forms of $v_d(X)$ are*

$$\mathbb{FF}^k_{v_d(X)} = \sum_{l_1+\ldots+l_d=k} c_{l_1\ldots l_d} F_{l_1}\ldots F_{l_d} \bmod \Big(\sum_{l<k} \mathbb{FF}^l_{v_d(X)}(S^lT)\Big)\Big|_{\ker \mathbb{FF}^{k-1}_{v_d(X)}},$$

where the $c_{l_1\ldots l_d}$ are nonzero constants.

For example,

$$\mathbb{FF}^4_{v_3(X)} = c_{400}F_4F_0F_0 + c_{310}F_3F_1F_0 + c_{220}F_2F_2F_0 + c_{211}F_2F_1F_1.$$

For the proof, see [**102**].

Remark 3.7.6. We need to be careful if we try to recover $F_k(X)$ from the $\mathbb{FF}^l_{v_d(X)}$, as cancellation could occur. For example, if X has small codimension, we will have $\mathbb{FF}^k_{v_2(X)} = 0$ for k large, but $F_k(X)$ will not be zero in general. We can recover F_k from $\mathbb{FF}^{k+1}_{v_{k-1}(X)}$, though.

If Z is a hypersurface of degree d, then Z osculates to X to order p at $x \in X$ if and only if $\vec{H}_Z \in \ker \mathbb{FF}^k_{v_d(X),x^d}$, where $\vec{H}_Z \in N^*_{v_d(x)}v_d(X)$ is the vector associated to the equation $H_Z \in \mathbb{P}S^dV^*$ of Z. Similarly, $X \subset Z$ iff $\vec{H}_Z \in \ker \mathbb{FF}^k_{v_d(X),x^d}$ for all k.

Proposition 3.7.7. [**102**] *Let $X^n \subseteq \mathbb{P}^{n+a}$ be a variety and let $x \in X_{\mathsf{smooth}}$. For all $p \leq d$, the dimension of the set of (not necessarily irreducible) hypersurfaces of degree d osculating to order p at x is*

$$\binom{n+a+d}{d} - \binom{n+p}{p}.$$

As with quadrics, for $p > d$, the dimensions of the spaces of osculating hypersurfaces depend on the geometry of X. We also see that, independent of X, for $d+1 \leq k \leq 2d-1$, there are *lower bounds* on the dimensions of the space of hypersurfaces of degree d osculating to order k at x. For example:

Proposition 3.7.8 ([**102**]). *Let $X^n \subseteq \mathbb{P}^{n+a}$ be a variety, and let $x \in X_{\mathsf{smooth}}$. The dimension of the set of (not necessarily irreducible) hypersurfaces of degree d osculating to order $2d-1$ at x is at least*

$$\binom{a+d-1}{d} - 1.$$

Higher-order Bertini Theorems. Let T be a vector space. The classical Bertini Theorem is as follows:

Theorem 3.7.9. *Let $A \subset S^dT^*$ be a linear subspace, let $q \in A$ be generic, and let $Q \subset \mathbb{P}T$ denote the hypersurface defined by $q = 0$. Then $v \in Q_{\mathsf{sing}}$ implies $v \in \mathrm{Baseloc}(A) := \{v \in T \mid v \in Q\ \forall Q \in A\}$.*

If A now varies in a family $\{A_s\}_{s \in S}$, there are restrictions on how A moves infinitesimally in the family. We will only be concerned with the case when $d = 2$ and the family is determined by the second fundamental form of a complex submanifold. In this case the higher-order differential invariants measure how the family varies. Although we use the language of submanifolds of projective space, these results are valid in the more general context of studying a subvariety $\Xi \subset G(k, S^2T^*)$.

Theorem 3.7.10 (Higher-order Bertini [**109**]). *Let $M^n \subset \mathbb{P}V$ be a complex submanifold and let $x \in M$ be a general point.*

1. Let $q \in |II|_{M,x}$ be a generic quadric. Then $Q_{\mathsf{sing}} \subset \mathrm{Baseloc}\{F_2, \dots, F_k\}$ for all k, i.e., Q_{sing} is tangent to a linear space on the completion of M.

2. Let $h \in N^*_xM$ and let $L \subset II(h)_{\mathsf{sing}} \cap \mathrm{Baseloc}\,|II|$ be a linear subspace. Then $h \lrcorner F_3(v, w, \cdot) = 0$ for all $v, w \in L$.

3. With h and L as in 2, if $L' \subset (\mathrm{Baseloc}\{|II|, F_3\}) \cap L$ is a linear subspace, then $h \lrcorner F_4(u, v, w, \cdot) = 0$ for all $u, v, w \in L'$, and so on for higher orders. (Note that $h \lrcorner F_4(u, v, w, \cdot)$ is well-defined by the lower-order vanishing.)

4. With h and L' as in 3, if $L'' \subset L' \cap (h \lrcorner F_3)_{\mathsf{sing}}$ is a linear space, then for all $u, v \in L''$ we have $h \lrcorner F_4(u, v, \cdot, \cdot) = 0$.

Proof. In each case, the extension to linear spaces will hold by polarizing the forms. We prove only the first assertion in each case, as an analogous equation at each order proves the next higher order.

1. Assume $v = \underline{e}_1$ and $q = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0$. Our hypotheses imply $q^\mu_{1\beta} = 0$ for all β. Formula (3.5) reduces to

$$r^\mu_{11\beta}\omega^\beta_0 = -q^\nu_{11}\omega^\mu_\nu.$$

If q is generic we are still working on $\mathcal{F}^1$, and so the ω^μ_ν are independent of the semi-basic forms; thus the coefficients on both sides of the equality are

zero. Moreover, the relevant equation for F_4 reduces to

$$r^{\mu}_{111\beta}\omega^{\beta}_{0} = -r^{\nu}_{111}\omega^{\mu}_{\nu},$$

and again, both sides are zero. This and the corresponding higher-order equations also prove both the classical Bertini Theorem and part 1 of the theorem.

2. If $h = \underline{e}^{n+a}$ is not generic, we have to reduce to a subbundle $\mathcal{F}' \subset \mathcal{F}^1$ where the ω^{n+a}_{ν} are not necessarily independent. However, the hypotheses of part 2 state that $q^{\nu}_{11} = 0$ for all ν, and the required vanishing still holds.

For part 3, note that $r^{n+a}_{111\delta}\omega^{\delta}_{0} = r^{\nu}_{111}\omega^{n+a}_{\nu} + r^{n+a}_{11\epsilon}\omega^{\epsilon}_{1} + q^{n+a}_{1\epsilon}q^{\nu}_{11}\omega^{\epsilon}_{\nu}$ and the right hand side is zero under our hypotheses. Part 4 is proven similarly.

□

3.8. Ruled and uniruled varieties

This section could be considered as a continuation of the previous subsection on higher-order Bertini theorems. We determine conditions on the differential invariants of M that imply it contains linear spaces through a general point.

Definition 3.8.1. Let $M \subset \mathbb{P}V$ be a submanifold or subvariety.

M is *ruled* by k-planes if it can be written as a fibration $M \to B$ where each fiber is (an open subset of) a k-plane.

M is *uniruled* by k-planes if for all $x \in M$ there exists at least one (open subset of a) k-plane L such that $x \in L \subset M$. We also sometimes say M is *$\mathbb{P}^k$-uniruled*.

M is *covered* by k-planes if M is $\mathbb{P}^k$-uniruled with just a finite number of k-planes passing through a general point.

Example 3.8.2. If $n \geq 4$, the quadric hypersurface $Q^n \subset \mathbb{CP}^{n+1}$ is uniruled but not ruled by $\lfloor \frac{n}{2} \rfloor$-planes.

Exercise 3.8.3: Show that a generic hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree n is covered by lines, with $n!$ lines passing through a general point.

We address the following question:

How many derivatives must one take to see if a submanifold is uniruled?

We saw in Chapter 2 that on any surface with $K < 0$ in $\mathbb{E}^3$ there are two lines osculating to order two at each point, so to see if a surface is ruled, we need to take at least three derivatives.

Theorem 3.8.4 (Darboux). *Let $M^2 \subset \mathbb{P}^{2+a}$ (or affine space) be a smooth (resp. analytic) surface such that at each (resp. a general) $x \in M$ there is a line l_x osculating to order three at x. Then M is ruled.*

Proof. We adapt frames such that e_1 is tangent to the line osculating to order three. Consider the integral curves C of the line field $\{\underline{e}_1\}$ (the annihilator of the system $\{\omega_0^2 = 0\}$). The second fundamental form of C is

$$II_{C,[e_0]} = \omega_1^2\omega_0^1 \otimes (e_2 \operatorname{mod} \hat{T}_{[e_0]}C) \otimes e^0 + \omega_1^\mu \omega_0^1 \otimes (e_\mu \operatorname{mod} \hat{T}_{[e_0]}C) \otimes e^0.$$

By hypothesis we have $\omega_1^\mu \equiv 0 \operatorname{mod}\{\omega_0^2\}$, and we need to calculate ω_1^2. Examining the formula (3.6) for F_3, we see that

$$r^\mu_{111}\omega_0^1 + r^\mu_{112}\omega_0^2 = 2q^\mu_{12}\omega_1^2.$$

If $II(\underline{e}_1, \underline{e}_2) = 0$ we are in the case of a degenerate Gauss map, and the result follows by Theorem 3.4.2. Otherwise there exists a μ, say $\mu = 3$, where $q^3_{12} \neq 0$. We use our third-order hypothesis, which implies $r^3_{111} = 0$, to conclude that $\omega_1^2 \equiv 0 \operatorname{mod}\{\omega_0^2\}$. Thus $II_{C,[e_0]} = 0$, so C is a line. $\square$

Remark 3.8.5. More generally, if $M^n \subset \mathbb{P}V$ is an analytic submanifold, then a line osculating to order $n+1$ at a general point must be contained in the completion of M; see [**106**].

Exercise 3.8.6: Let $M^2 \subset \mathbb{P}^3$ be covered by lines. Show that there are at most two lines passing through a general point. The same result holds in arbitrary codimension, as lines map to lines under a linear projection.

Let $\Sigma_x^k \subset \mathbb{P}T_xX$ denote the variety of tangent directions to lines osculating to order k at x. If X is a hypersurface, Σ^k is the zero set of polynomials of degrees $2, 3, \dots, k$, so one would expect Σ^{n+1} to be empty. Remark 3.8.5 above says that if it is not, it consists of tangent directions to actual lines. More generally, the expected dimension of Σ^k is $n-k$, and we have the following result:

Theorem 3.8.7 ([**108**]). *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface and let $x \in X$ be a general point. If there is an irreducible component $\Sigma_0^k \subset \Sigma^k$ such that $\dim \Sigma_0^k > n-k$, then all lines in $\mathbb{P}V$ through x, tangent to directions in Σ_0^k, are contained in X.*

Proof. We choose a basis $e_1, \dots, e_n$ of T_xX such $[e_1]$ is a general point of Σ_0^k, and $\tilde{T}_{[e_1]}\Sigma^k = \mathbb{P}\{e_1, e_2, \dots, e_p\}$, where $p - 1 = \dim \Sigma_0^k > n-k$.

Let $1 \leq \alpha, \beta \leq n$. In this proof only, we change notation slightly and let $r_{\alpha,\beta}$ denote the coefficients of the second fundamental form II of X at x, while $r_{\alpha_1,\dots,\alpha_i}$ denotes the coefficients of F_i. We write $r^i_{\alpha_1,\dots,\alpha_i} = r_{\alpha_1,\dots,\alpha_i}$ to keep track of i. We use index ranges: $2 \leq s,t \leq p$ and $p+1 \leq j,l \leq n$.

By our normalizations, we have $r^{\lambda}_{1,\dots,1} = 0$ and $r^{\lambda}_{1,\dots,1,s} = 0$ for $2 \leq s \leq p$ and $2 \leq \lambda \leq k$. We will show that $r^{h}_{1,\dots,1} = 0$ and $r^{h}_{1,\dots,1,s} = 0$ for $2 \leq s \leq p$ and for all h, which will imply that there is a p-dimensional space of lines passing through X at x.

We proceed by induction on h, using our lower-order equations to solve for the connection forms ω^j_1 in terms of the semi-basic forms ω^l_0, and then plug into the higher F_h to show that $F_h(e_1, \dots, e_1)$ and $F_h(e_1, \dots, e_1, e_s)$ both vanish. Using (3.7) we obtain that, for $\lambda \leq k$,

$$r^{\lambda}_{1,\dots,1,i}\omega^i_0 = -r^{\lambda-1}_{1,\dots,1,j}\omega^j_1.$$

These are $k-2$ equations for the $n-p$ one-forms ω^j_1 in terms of the $n-p$ semi-basic forms ω^j_0. Recall that $n - p \leq k - 2$. The system is solvable, because were the $(k-2) \times (n-p)$ matrix $(r^{\lambda}_{1,\dots,1,j})$ not of maximal rank, there would be additional directions in the tangent space to Σ^k_0. Thus we have

$$\omega^j_1 \equiv 0 \operatorname{mod}\{\omega^{p+1}_0, \dots, \omega^n_0\},$$

so the equation

$$r^{k+1}_{1,\dots,1,\beta}\omega^{\beta}_0 = -r^{k}_{1,\dots,1,j}\omega^j_1$$

implies $r^{k+1}_{1,\dots,1} = r^{k+1}_{1,\dots,1,2} = \dots = r^{k+1}_{1,\dots,1,p} = 0$. Thus the line through $[e_1]$ has contact to order $k+1$, and moreover $\tilde{T}_{[e_1]}\Sigma^{k+1} = \tilde{T}_{[e_1]}\Sigma^k$. Now one can use these equations iteratively to show the same holds to order $k+2$ and all orders, i.e., the component of Σ^k containing $[e_1]$ equals the component of Σ^{∞} containing $[e_1]$. □

We obtain the following corollary:

Theorem 3.8.8 ([**108**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be covered by lines. Then there are at most $n!$ lines passing through a general point of X.*

Proof. Without loss of generality we may assume X is a hypersurface, as one can reduce to this case by linear projection. First note that $n!$ is the expected bound in the sense that in $\mathbb{P}T_xX$ one has the ideal I generated by $F_2, F_3, \dots, F_n$ defining the variety $\Sigma^n \subset \mathbb{P}T_xX$ of all lines having contact with X at x to order n. (The polynomials of degree greater than two are not well-defined individually, but the ideal I is.) Since $\mathbb{P}T_xX$ is a $\mathbb{P}^{n-1}$, if Σ^n is zero-dimensional, it is at most $n!$ points and we are done. If $\dim \Sigma^n > 0$, then Theorem 3.8.7 applies. □

3.9. Varieties with vanishing Fubini cubic

We saw in Exercise 3.2.12.2 that if the second fundamental form of a submanifold M is identically zero, then the completion of M is a linear space. What is the geometric interpretation of $F_3 = 0$? First of all, the condition

must be correctly interpreted–that there exists a framing of M, $s : M \to \mathcal{F}^1$ such that $s^*(F_3) = 0$. In codimension one we have the following theorem:

Theorem 3.9.1 (Fubini [**50**]). *Let $M^n \subset \mathbb{CP}^{n+1}$, $n \geq 2$, be an algebraic or analytic hypersurface with γ_M nondegenerate. If there exists a framing $s : M \to \mathcal{F}^1$ such that $s^*(F_3) = 0$, i.e., if for all sections $s : M \to \mathcal{F}^1$ we have $s^*(F_3) = l \circ II$ with $l \in \Omega^1(M)$, then M is (an open subset of) the smooth quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$.*

Remark 3.9.2. Fubini's Theorem is valid over $\mathbb{R}$ and in the C^∞ category, with the conclusion that the hypersurface is an open subset of the quadric whose second fundamental form has the same signature as M's. (The same proof as below works, although there are several cases.)

We first study $SO(V,Q) \to Q$ as an adapted frame bundle. Use linear coordinates $(x^0, \ldots, x^{n+1})$ on V and let $Q = \{x^0 x^{n+1} - \sum_\alpha (x^\alpha)^2 = 0\}$. We use Q to denote both the quadratic form and the quadric hypersurface it determines.

Exercise 3.9.3: Show that, with this choice of basis,

$$\mathfrak{so}(V,Q) = \left\{ \begin{pmatrix} a^0_0 & a^0_\beta & 0 \\ a^\alpha_0 & a^\alpha_\beta & a^\alpha_{n+1} \\ 0 & a^{n+1}_\beta & a^{n+1}_{n+1} \end{pmatrix} \;\middle|\; \begin{array}{ll} a^0_0 + a^{n+1}_{n+1} = 0, & a^{n+1}_\beta - a^\beta_0 = 0, \\ a^\alpha_\beta + a^\beta_\alpha = 0, & a^\alpha_{n+1} - a^0_\alpha = 0 \end{array} \right\}.$$

Note that with these choices, and the projection

$$SO(V,Q) \to \mathbb{P}V,$$
$$(e_0, \ldots, e_{n+1}) \mapsto [e_0],$$

$SO(V,Q)$ is a bundle of first-order adapted frames to Q.

Exercise 3.9.4: Show that in these frames, $II_{Q,[e_0]} = ((\omega^1_0)^2 + \ldots + (\omega^n_0)^2) \otimes \underline{e}_{n+1}$ and $F_3 = 0$, i.e., that if $i : SO(V,Q) \subset \mathcal{F}^1_Q$ is the inclusion, then $i^*(F_3) = 0$.

Proof of 3.9.1. We now begin with M an unknown hypersurface such that $F_3 = 0$ in some framing, and show that a subbundle of the bundle of frames f where $F_{3f} = 0$ and $\det(f) = 1$ is $SO(V,Q)$. We then use Theorem 1.6.10 to conclude the proof.

We normalize the second fundamental form so that

$$\omega^{n+1}_\alpha = \omega^\alpha_0$$

and reduce to the subbundle of the frame bundle where $\det(f) = 1$. Then

$$0 = r_{\alpha\beta\gamma}\omega^\gamma_0 = -\delta_{\alpha\beta}(\omega^0_0 + \omega^{n+1}_{n+1}) + (\omega^\alpha_\beta + \omega^\beta_\alpha),$$

which shows that

$$\begin{aligned} \omega^\alpha_\beta + \omega^\beta_\alpha &= 0 \qquad\qquad \text{for } \alpha \neq \beta, \\ 2\omega^\alpha_\alpha &= \omega^0_0 + \omega^{n+1}_{n+1} \quad \forall \alpha. \end{aligned}$$

The second equation implies that $\omega^\alpha_\alpha = \omega^\beta_\beta$ for all α, β, and the condition $\det(f) = 1$ implies $\mathrm{trace}(\omega) = 0$, so

$$\begin{aligned} (n+2)\omega^\alpha_\alpha &= \omega^0_0 + \omega^{n+1}_{n+1} + \omega^1_1 + \ldots + \omega^n_n \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \omega^\alpha_\alpha &= 0, \ \forall \alpha, \\ \omega^0_0 + \omega^{n+1}_{n+1} &= 0. \end{aligned}$$

To reduce the Maurer-Cartan form to the Maurer-Cartan form for $SO(V)$, we need to show that $\omega^\alpha_{n+1} - \omega^0_\alpha = 0$ and $\omega^0_{n+1} = 0$. We consider the coefficients of F_4:

$$\begin{aligned} r_{\alpha\beta\gamma\epsilon}\omega^\epsilon_0 &= 0, && \alpha, \beta, \gamma \text{ distinct}, \\ r_{\alpha\alpha\beta\epsilon}\omega^\epsilon_0 &= (\omega^0_\beta - \omega^\beta_{n+1}), && \alpha \neq \beta, \\ r_{\alpha\alpha\alpha\epsilon}\omega^\epsilon_0 &= 3(\omega^0_\alpha - \omega^\alpha_{n+1}). \end{aligned}$$

We see that for all $\alpha \neq \beta$, we have $r_{\alpha\alpha\alpha\beta} = 0, r_{\alpha\alpha\beta\beta} = r$ and $r_{\alpha\alpha\alpha\alpha} = 3r$ for some function r.

Exercise 3.9.5: Show that we may reduce our frame bundle further to a subbundle where $r \equiv 0$.

Now that all coefficients of F_3 and F_4 are zero, we see that F_5 and all higher F_k must be zero as well, as, for example, the coefficients of F_5 are polynomials in the coefficients of II, F_3, F_4, but the coefficients of II only occur multiplied by the coefficients of F_3. □

Exercise 3.9.6: Show that Fubini's Theorem still holds for manifolds with degenerate Gauss maps as long as the rank of the Gauss map is at least two. In this case the closure of M is a singular quadric hypersurface.

Exercise 3.9.7: Show that, for $G(k, W) \subset \mathbb{P}(\Lambda^k W)$ and $\mathrm{Seg}(\mathbb{P}^k \times \mathbb{P}^l)$, there are natural framings with $F_3 = 0$.

Remark 3.9.8. Not all homogeneous varieties admit a framing where $F_3 = 0$. For example, the space of traceless and rank one $n \times n$ matrices, as a subvariety of the space of all matrices, does not admit a framing where $F_3 = 0$; see [**110**]. The generalized minuscule varieties do admit framings where $F_3 = 0$ and are the unique homogeneous varieties having this property (see [**110, 111**]). There are examples of inhomogeneous varieties that have

this property, although such varieties are locally homogeneous, e.g., the completion to a projective variety of a paraboloid given in local coordinates as $x^\mu = q^\mu_{\alpha\beta}x^\alpha x^\beta$.

Exercise 3.9.9: Let $X^n \subset \mathbb{P}W$ be a variety with $|II| = S^2T^*_xX$ and such that there exists a framing in which $F_3 = 0$ for general $x \in X$. Show that $X = v_2(\mathbb{P}^n)$. ⊚

3.10. Dual varieties

We now study varieties $X \subset \mathbb{P}V$ with degenerate dual varieties $X^* \subset \mathbb{P}V^*$. We begin by finding a frame bundle simultaneously adapted to the first-order geometry on X and X^*. We then prove the basic theorems about dual varieties: reflexivity, the Landman parity theorem and Zak's bound on the dual defect.

Frames for X and X^*. Consider the incidence correspondence (3.3). As usual, we work on a frame bundle, using the geometry of the incidence correspondence to guide us. Let $\mathcal{F}^{*\prime} \to \mathcal{I}^0 := \{(x, H) \mid x \in X_{\mathsf{smooth}}, H \in X^*_{\mathsf{smooth}}, \tilde{T}_xX \subseteq H\}$ be the frame bundle of bases of V whose fiber over $(x, H) \in \mathcal{I}^0$ consists of bases adapted to the flag

$$0 \subset \hat{x} \subset \hat{T}_xX \subset \hat{H}^\perp \subset V,$$

that is, bases $(e_0, \dots, e_{n+a})$ such that

$$\begin{aligned}\{e_0\} &= \hat{x},\\ \{e_0, \dots, e_n\} &= \hat{T}_xX,\\ \{e_0, \dots, e_{n+a-1}\} &= \hat{H}.\end{aligned}$$

Let $\pi' : \mathcal{F}^{*\prime} \to \mathbb{P}V$ and $\rho' : \mathcal{F}^{*\prime} \to \mathbb{P}V^*$ denote the projections, so $\pi'(e_0, \dots, e_{n+a}) = [e_0]$ and $\rho'(e_0, \dots, e_{n+a}) = [e^{n+a}]$, where $(e^0, \dots, e^{n+a})$ is the basis dual to $(e_0, \dots, e_{n+a})$.

Then

$$X = \overline{\pi'(\mathcal{F}^{*\prime})}, \qquad X^* = \overline{\rho'(\mathcal{F}^{*\prime})},$$

and thus if $f \in \mathcal{F}^{*\prime}$ is a general point, then $\dim(X^*) = \operatorname{rank} \rho'_{*f}$.

To differentiate vectors in the dual frame $(e^0, \dots, e^{n+a})$, we need to see how the Maurer-Cartan form behaves with respect to dual bases. Let $\langle , \rangle$ denote the pairing $V \times V^* \to \mathbb{C}$, with $\langle e_A, e^B \rangle = \delta^B_A$. We calculate

$$0 = d\langle e_A, e^B \rangle = \langle \omega^C_A e_C, e^B \rangle + \langle e_A, de^B \rangle,$$

which implies

$$\langle de^B, e_A \rangle = -\omega^B_A,$$

i.e.,

$$de^B = -\omega^B_C e^C.$$

Since ρ is submersive, we may calculate $\dim X^*$ by determining the number of independent forms in de^{n+a}:

$$de^{n+a} \equiv -\omega^{n+a}_\alpha e^\alpha - \omega^{n+a}_{n+\lambda} e^{n+\lambda} \operatorname{mod} e^{n+a}. \tag{3.15}$$

Here we use the index range $1 \leq \lambda, \kappa \leq a-1$. Note that the forms $\omega^{n+a}_{n+\lambda}$ are independent and independent of the semi-basic forms for π', while $\dim\{\omega^{n+a}_\alpha\} = \operatorname{rank} q^{n+a}$. We obtain an explicit description of the tangent space:

Theorem 3.10.1. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety, let $x \in X$ be a general point and let $H \in X^*$ correspond to a general point $\vec{H} \in N^*_x X$. Then*

$$\hat{T}_H X^* = (\hat{x} + \widehat{\operatorname{Singloc} II(\vec{H})})^\perp,$$

where $\widehat{\operatorname{Singloc} II(\vec{H})}$ is considered as a subset of $\hat{T}_x X/\hat{x}$ (i.e., we ignore the twist by a line bundle), so the sum makes sense as a subspace of V.

Corollary 3.10.2 ([**68**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety, let $x \in X$ be a general point and let r be the rank of a generic quadric in $|II_{X,x}|$. Then $\dim X^* = r + a - 1$.*

Exercise 3.10.3: Calculate $\dim X^*$ for the following varieties:

(a) X is a curve in $\mathbb{P}^{a+1}$.
(b) $X = v_d(\mathbb{P}V)$.
(c) $X = \operatorname{Seg}(\mathbb{P}^a \times \mathbb{P}^b)$.
(d) $X = G(2, m)$.
(e) X is a smooth surface in $\mathbb{P}^{a+2}$, not contained in a hyperplane.

Which of these are self-dual (i.e., isomorphic to their dual varieties)? ◎

Corollary 3.10.4. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety, with dual variety X^* of dimension $r + a - 1$, let $x \in X$ be a general point and let $\vec{H} \in N^*_x X$ correspond to a point $H \in X^*$. If $\operatorname{rank} II(\vec{H}) < r$, then $H \in X^*_{\mathsf{sing}}$.*

Proof. Equation (3.15) shows that if $\operatorname{rank} II(\vec{H}) < r$, then $\hat{T}_{[e^{n+a}]} X^*$ has the wrong dimension. □

Observe that Theorem 3.10.1 implies $[e_0] \in \mathbb{P}N^*_H X^*$; thus, if $x \in X_{\mathsf{smooth}}$ and $H \in X^*_{\mathsf{smooth}}$, then

$$\tilde{T}_x X \subseteq H \Leftrightarrow \tilde{T}_H X^* \subseteq x.$$

Note that $\mathcal{I}^0$ surjects onto Zariski open subsets of X and X^*, so taking closures we conclude:

Theorem 3.10.5 (Reflexivity[1]). $(X^*)^* = X$.

[1]The reflexivity theorem has a long history, dating back to C. Segre; see [**91**] for details.

Thus, the dual variety should be viewed as a transform of X, containing the same information as X, only reorganized. We will see (in Theorem 3.10.16 below) that in this re-organization, some global information becomes local.

Definition 3.10.6. Given $X^n \subset \mathbb{P}^{n+a}$, define $\delta_* = \delta_*(X) = n + a - 1 - \dim X^*$, the *dual defect* of X.

The reflexivity theorem implies that $\mathbb{P}N_H^* X^* \subset X$, and thus we have

Proposition 3.10.7. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety, let $x \in X$ be a general point, and let $H \in X^*$ be a smooth point with $\tilde{T}_x X \subseteq H$. Then*

$$\pi(\rho^{-1}(H)) = \{y \in X \mid \tilde{T}_y X \subseteq H\}$$

is a linear $\mathbb{P}^{\delta_*} \subset X$ and may be identified with $\mathbb{P}N_H^* X^*$.

Reversing the roles of X and X^* in Corollary 3.10.4, we obtain:

Theorem 3.10.8. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, let $H \in X^*$ be a general point, let $v \in N_H^* X^*$ be any nonzero vector, and let $r = \operatorname{rank} II_{X^*,H}(v)$. Then $\dim(X^*) = r + a - 1$. In particular, $|II_{X^*,H}|$ is a system of quadrics of projective dimension equal to the dual defect δ_*, and of constant rank $n - \delta_*$.*

Theorem 3.10.8 is evidently due to B. Segre [**136**]. It was rediscovered by Griffiths and Harris in [**69**].

Corollary 3.10.9. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, and let $H \in X^*$ be a general point. Then $III_{X^*,H} = 0$.*

It is difficult to have a system of quadrics of constant rank. In fact, by a classical result in linear algebra, it is impossible if r is odd, and, by a more recent result [**83**], impossible if the dimension of the system is one greater than the dimension of the vector space minus the rank. Thus we unify and recover two important results:

Theorem 3.10.10 (Landman parity theorem [**44**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety. Then $n - \delta_*$ is even.*

Theorem 3.10.11 (Zak [**157**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety not contained in a hyperplane. Then $\delta_* < a - 1$, i.e., $\dim X^* \geq \dim X$.*

One consequence of Zak's theorem is the following:

Theorem 3.10.12 ([**100**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, let $x \in X$ be a general point and let $q \in |II_{X,x}|$ be a generic quadric. Then $\dim(\operatorname{Singloc}(q)) \leq a - 1$.*

We will generalize Theorem 3.10.12 in §3.14.

Detour: Systems of quadrics of constant rank. Let $V = \mathbb{C}^m$ and let $k > 1$. What are the k-dimensional linear subspaces of S^2V of constant rank r?

We saw above that r is necessarily even. When $k = 2$ there is a complete answer:

Normal form for pencils of quadrics of constant rank. Let

$$A_p[s,t] = \begin{pmatrix} s & 0 & 0 & \dots & 0 \\ t & s & 0 & \dots & 0 \\ 0 & t & s & 0 & \dots \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & t & s \\ 0 & \dots & & 0 & t \end{pmatrix}$$

be a pencil of $\frac{p}{2} \times (\frac{p}{2}+1)$ matrices. Note that for all $[s,t] \in \mathbb{P}^1$, $\operatorname{rank} A_p[s,t] = \frac{p}{2}$. Let

$$Q_p[s,t] = \begin{pmatrix} 0 & A_p[s,t] \\ {}^tA_p[s,t] & 0 \end{pmatrix}$$

and note that it is a $(p+1)\times(p+1)$ *pencil*, i.e., a two-dimensional linear subspace of quadrics, of constant rank p. Any constant rank p pencil of quadrics on $\mathbb{C}^{p+1}$ can be normalized to be of this form. Note that $\operatorname{Singloc} Q_p[s,t]$ is a rational normal curve of degree $\frac{p}{2}$ in the $\mathbb{P}^{\frac{p}{2}}$ spanned by the first $\frac{p}{2}+1$ directions, and its linear span distinguishes a $\mathbb{C}^{\frac{p}{2}+1} \subset \mathbb{C}^{p+1}$. Thanks to the SL_2-action, there is a complementary $\mathbb{C}^{\frac{p}{2}} \subset \mathbb{C}^{p+1}$ that is also distinguished.

From [**79**], any constant rank r pencil of quadratic forms on $\mathbb{C}^m$ is of the form

$$Q[s,t] = \begin{pmatrix} Q_{i_1}[s,t] & & & \\ & \ddots & & \\ & & Q_{i_l}[s,t] & \\ & & & 0 \end{pmatrix},$$

where if the last zero block is $f^* \times f^*$, we have $i_1 + \ldots + i_l = r$, $l + f^* = m - r$. In our applications we will have $m = r + a - 1$.

The classification problem can be studied systematically using vector bundles on projective space; see [**83, 107**].

Remark 3.10.13. For those familiar with the language of vector bundles, given a system of quadrics $A \subset \mathbb{P}S^2V^*$ of constant rank r, we obtain a rank r vector bundle $E \to \mathbb{P}A$ whose fiber at $[q] \in \mathbb{P}A$ is the image of the linear map $q : V \to V^*$. Muñoz [**123**] observed that an argument of Ein in [**45**] (used in the proof of his linear fibration theorem) actually shows that if $\operatorname{rank} E > 1$ and E is the direct sum of line bundles, then any X with $|II_X| \simeq A$ at general points is ruled by linear spaces. Sato [**135**] showed

that if E is a uniform vector bundle and $\operatorname{rank} E < a$, then in this case E must be the direct sum of line bundles. Using Sato's result, Ein [**45**] proved that if $\delta_* \geq \frac{n}{2}$, then $X^n \subset \mathbb{P}^{n+a}$ must be $\mathbb{P}^{(n+\delta_*)/2}$-ruled, which is his linear fibration theorem.

Exercises 3.10.14:

1. Show that from a linear space B of $p \times q$ matrices of constant rank s, one can construct a system of quadrics of constant rank $r = 2s$ on $\mathbb{C}^m$ with $m = p + q$. ⊚
2. Use exercise 1 to construct $|II_{G(2,5)}|$. ⊚
3. One gets the second fundamental form of what variety when one uses exercise 1 in the case $B = M_{2\times q}$? ⊚

Calculation of $II_{X^*,H}$. We now reduce our frame bundle to adapt to the distinguished subspace $\operatorname{Singloc}(q^{n+a})$: Let $\mathcal{F}^* \to \mathcal{I}^0$ be the frame bundle of bases of V over $(x, H) \in \mathcal{I}^0$ adapted to the flag

$$0 \subset \hat{x} \subset \{\hat{x}, \hat{q}^H_{\mathsf{sing}}\} \subset \hat{T}_x X \subset \hat{H}^\perp \subset V,$$

where $q_H \in \mathbb{P}S^2T^*_x X$ is the quadric corresponding to $\vec{H} \in N^*_x X$. That is, $\mathcal{F}^*$ is the bundle of frames $(e_0, \dots, e_{n+a})$ such that

$$\begin{aligned}
\{e_0\} &= \hat{x}, \\
\{e_0, \dots, e_{n-r}\} &= \hat{x} + \operatorname{Singloc}(q^H_x), \\
\{e_0, \dots, e_n\} &= \hat{T}_x X, \\
\{e_0, \dots, e_{n+a-1}\} &= \hat{H}.
\end{aligned}$$

Observe that $\mathcal{F}^*$ is also first-order adapted to X^* because $\{\hat{x}, \hat{q}^H_{\mathsf{sing}}\} = (\hat{T}_H X^*)^\perp$. The dual flag is

$$0 \subset \hat{H} \subset (\hat{T}_x X)^\perp \subset \hat{T}_H X^* \subset \hat{x}^\perp \subset V^*.$$

Applying Theorem 3.10.1 with the roles of X and X^* reversed, we see that $(\hat{T}_x X)^\perp = \{\hat{H}, II_{X^*,H}(\vec{x})\}$, so $\mathcal{F}^*$ is equally adapted over X and X^*.

Fix further index ranges $1 \leq s, t, u \leq n - r$, $n - r + 1 \leq i, j, k \leq n$.

Write the pullback of the Maurer-Cartan form to $\mathcal{F}^*$ as

$$\Omega = \begin{pmatrix}
\omega^0_0 & \omega^0_t & \omega^0_k & \omega^0_{n+\lambda} & \omega^0_{n+a} \\
\omega^s_0 & \omega^s_k & \omega^s_t & \omega^s_{n+\lambda} & \omega^s_{n+a} \\
\omega^j_0 & \omega^j_k & \omega^j_t & \omega^j_{n+\lambda} & \omega^j_{n+a} \\
0 & \omega^{n+\kappa}_t & \omega^{n+\kappa}_k & \omega^{n+\kappa}_{n+\lambda} & \omega^{n+\kappa}_{n+a} \\
0 & 0 & \omega^{n+a}_k & \omega^{n+a}_{n+\lambda} & \omega^{n+a}_{n+a}
\end{pmatrix},$$

where $\omega^{n+a}_t = 0$ as we have adapted frames so that $\{\underline{e}_t\} = \operatorname{Singloc}(q^{n+a})$.

We calculate

$$de^{n+a} \equiv -\omega_k^{n+a} e^k - \omega_{n+\lambda}^{n+a} e^{n+\lambda} \operatorname{mod} e^{n+a}.$$

Note that $\omega_k^{n+a} = q_{kj}^{n+a}\omega_0^j$ with q_{kj}^{n+a} invertible, and the forms $\omega_{n+\lambda}^{n+a}$ are all independent and independent of the semi-basic forms for π.

Exercise 3.10.15: Give a proof of Proposition 3.10.7 using a naïve frame argument as we did for varieties with degenerate Gauss maps. Namely, fix $H = [e^{n+a}]$, and show that the distribution $\hat{q}^{n+a}|_{\mathsf{sing}} = \{e_0, e_s\}$ is integrable and that the integral manifold Y containing $[e_0]$ has $II_Y = 0$.

We now calculate $|II_{X^*,H}|$. First note that $q_{st}^{\mu} = 0$ for all μ, s, t by Proposition 3.7.9. To compute $II_{X^*,H}$ we calculate

$$\begin{aligned} de^k &\equiv \omega_s^k e^s + \omega_0^k e^0 \operatorname{mod} \hat{T}_H X^*, \\ de^{n+\lambda} &\equiv \omega_s^{n+\lambda} e^s \qquad \operatorname{mod} \hat{T}_H X^*. \end{aligned}$$

In our situation, (3.5) yields

$$\begin{aligned} r_{st\beta}^{n+a}\omega_0^{\beta} &= 0, \\ r_{sij}^{n+a}\omega_0^j &= -q_{si}^{n+\lambda}\omega_{n+\lambda}^{n+a} + q_{kj}^{n+a}\omega_s^k, \end{aligned}$$

which implies

$$II_{X^*,H} = (r_{sij}^{n+a}\omega_0^j + 2q_{si}^{n+\lambda}\omega_{n+\lambda}^{n+a})\omega_0^i e^s + q_{jk}^{n+a}\omega_0^j\omega_0^k e^0 \operatorname{mod} \hat{T}_H X^*.$$

Observe that if $H = [e^{n+a}] \in X^*_{\mathsf{smooth}}$ then de^{n+a} has rank $r + a - 1$. Since the forms $\omega_{n+\lambda}^{n+a}$ are all independent and independent of the semi-basic forms, we recover the fact that the forms $\{\omega_j^{n+a}\}$ must span an r-dimensional space, i.e. $II(\vec{H})$ has rank r.

Proposition 3.10.16 (Inversion formula [**83**]). *With respect to the bases* $(\omega_{n+\lambda}^{n+a}, \omega_0^j)$ *of* $T_H^*X^*$, *and* (e^0, e^s) *of* $N_H X^*$, *we have*

$$II_{X^*,H} = \left\{ Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & q_{jk}^{n+a} \end{pmatrix}, \ Q_s = \begin{pmatrix} 0 & q_{sj}^{n+\lambda} \\ q_{sk}^{n+\lambda} & r_{sjk}^{n+a} \end{pmatrix} \right\},$$

where the blocking is $(a-1, r) \times (a-1, r)$.

Remark 3.10.17. For further reading on dual varieties, see the forthcoming monograph [**148**].

3.11. Associated varieties

In this section we generalize the notion of dual variety and Gauss image. Let $X^n \subset \mathbb{P}V = \mathbb{P}^{n+a}$ be a variety. Define the (r, s)-*associated variety of* X

to be

$$Z_{r,s}(X) = \overline{\{E \in G(r+1,V) \mid \exists x \in X_{\mathsf{smooth}} \text{ such that } \dim(\hat{T}_x X \cap E) \geq s+1\}}$$

Note that $Z_{n,n}(X) = \gamma(X)$ and $Z_{n+a-1,n} = X^*$, and for $r \geq a$ we have $Z_{r,0}(X) = G(r+1,V)$. We will see below that $Z_{a-1,0}(X)$ is always a hypersurface, and we expect $Z_{s+a-1,s}(X)$ to be a hypersurface. $Z_{a-1,0}(X)$ is called the *associated hypersurface* to X in **[60]** and the varieties $Z_{s+a-1,s}(X)$ are called *higher associated hypersurfaces*, although they are not always hypersurfaces—see below. Let $\delta_s(X) = \operatorname{codim} Z_{s+a-1,s}(X)$ and recall the notation $\delta_*(X) = \delta_n(X)$.

Proposition 3.11.1. $\delta_s(X) = \max\{0, s + \delta_*(X) - n\}$.

Proof. Fix index ranges $1 \leq \sigma, \tau \leq s$, $s+1 \leq j \leq n$, $1 \leq \lambda \leq a$ and let $f = (e_0, \ldots, e_{n+a})$ be a general point of $\mathcal{F}^1(X)$. Write

$$\vec{E} = e_0 \wedge e_1 \wedge \ldots \wedge e_s \wedge e_{n+1} \wedge \ldots \wedge e_{n+a-1}.$$

Then $E = [\vec{E}]$ is a general point of $Z_{s+a-1,s}(X)$. We calculate the dimension of $Z_{s+a-1,s}(X)$ by calculating the dimension of its tangent space at a general point. As before, let $\vec{E}^A_B$ be the vector with e_A removed and e_B put in its place in the wedge product. Then

(3.16)
$$d\vec{E} \equiv \vec{E}^0_j \omega^j_0 + \vec{E}^\sigma_j \omega^j_\sigma + \vec{E}^\sigma_{n+a} \omega^{n+a}_\sigma + \vec{E}^{n+\lambda}_j \omega^j_{n+\lambda} + \vec{E}^{n+\lambda}_{n+a} \omega^{n+a}_{n+\lambda} \mod \vec{E}.$$

The forms $\omega^j_0, \omega^j_\sigma, \omega^{n+a}_{n+\lambda}$ are all independent of each other, but $\omega^{n+a}_\sigma = q^{n+a}_{\sigma\tau}\omega^\tau_0 + q^{n+a}_{\sigma k}\omega^k_0$. $Z_{s+a-1,s}(X)$ will be a hypersurface iff the matrix $q^{n+a}_{\sigma\tau}$ is of full rank. Since this is the restriction of a generic quadric in the second fundamental form to a generic subspace, we fail to have full rank iff $\delta_*(X)$ (which is n minus the rank of a generic quadric in $|II|$) is greater than $n-s$ and the difference, if positive, is exactly $s + \delta_*(X) - n$. In particular, we see that $Z_{a-1,0}(X)$ is always a hypersurface. □

Remark 3.11.2. In **[60]**, $Z_{s+a-1,s}(X)$ is denoted by $Z_{n-s+1}(X)$ and in Proposition 3.2.11 of **[60]** (p. 104) it is mistakenly asserted that $Z_{n-s+1}(X)$ is always a hypersurface.

Given a hypersurface $Y \subset G(k,V)$ and $E \in Y_{\mathsf{smooth}}$, we have $\mathbb{P}N^*_E Y \in \mathbb{P}(E \otimes (V/E)^*)$. We define the *rank* of $N^*_E Y$ to be the rank of a general point of $\mathbb{P}N^*_E Y$. If the rank is one, i.e., $\mathbb{P}N^*_E Y \in \operatorname{Seg}(\mathbb{P}E \otimes \mathbb{P}(V/E)^*)$, then Y is said to be *coisotropic*.

Proposition 3.11.3 ([**60**], 4.3.14). *Let $X \subset \mathbb{P}V$ be a variety. If $Y = Z_{s+a-1,s}(X)$ is a hypersurface, then it is coisotropic.*

Proof. Taking E as above (and assuming Y is a hypersurface), the only vector that does not appear in $d\vec{E}$ is $\vec{E}^0_{n+a}\omega_0^{n+a}$, which is of rank one. □

Proposition 3.11.4 ([**60**], 4.3.14). *The coisotropic hypersurfaces in $G(k,V)$ are exactly the codimension one associated varieties to varieties $X \subset \mathbb{P}(V)$.*

Exercise 3.11.5: Prove the proposition by imitating our proof for Gauss images. Namely, take the natural variety X formed from Z and show that Z is an associated variety to X. You can use the calculation (3.16).

3.12. More on varieties with degenerate Gauss maps

Throughout this section, $X^n \subset \mathbb{P}V$ is a variety with degenerate Gauss map with f dimensional fibers. We now study such varieties in some more detail and take advantage of the cubic form. We keep the index ranges of §3.4; in particular, we are working on $\mathcal{F}^\gamma \subset \mathcal{F}^1$ where $q^\mu_{st} = q^\mu_{sj} = 0$. We have

$$\begin{aligned} r^\mu_{st\beta} &= 0 \qquad && \forall \mu, s, t, \beta, \\ r^\mu_{sij}\omega_0^j &= q^\mu_{jk}\omega^k_s \qquad && \forall \mu, s, i. \end{aligned} \tag{3.17}$$

Recall that we obtain the second fundamental form at $[e_0]$ by calculating $de_0 \equiv \omega_0^\alpha e_\alpha \operatorname{mod} e_0$, $de_\alpha \equiv \omega^\mu_\alpha e_\mu \operatorname{mod} \hat{T}_{[e_0]}X$, and writing $II = \omega_0^\alpha\omega^\mu_\alpha \underline{e}_\mu$, where the ω_0^α are the coefficients of e_α in de_0 and the ω^μ_α are the coefficients of e_μ in de_α. We have adapted frames such that $[e_s] \in X$. We now restrict to the open subset of $\mathcal{F}^\gamma$ where each $[e_s] \in X_{\mathsf{smooth}}$. We have

$$\begin{aligned} de_s &\equiv e_j\omega^j_s \qquad && \operatorname{mod} e_s, \\ de_j &\equiv e_\mu\omega^\mu_j \qquad && \operatorname{mod} \hat{T}_{[e_s]}X. \end{aligned}$$

Combining this with (3.17) we obtain

Proposition 3.12.1. *Let $x \in X$ and let F denote the fiber of γ passing through x. Then we may recover the second fundamental form of X at all $y \in F \cap X_{\mathsf{smooth}}$ from $II_{X,x}$ and $F_{3,X,x}$. More precisely, we have the formula*

$$II_{X,[e_s]} = \omega^\mu_j\omega^j_s(e_\mu \operatorname{mod} \hat{T}_{[e_s]}X)\otimes e^s = r^\mu_{sik}\omega^i_0\omega^k_0(e_\mu \operatorname{mod} \hat{T}_{[e_s]}X)\otimes e^s,$$

where there is no sum on s on the right hand side.

We saw several examples of varieties with degenerate Gauss mappings in §3.4. It would be nice to be able to identify them in terms of the quantities we have calculated. The easiest case is that of cones.

Characterization of cones over $\mathbb{P}^{f-1}$'s.

Proposition 3.12.2. *Assume $f < n-1$. Then X is a cone over a variety with a nondegenerate Gauss map iff for all $x, y \in F \cap X_{\mathsf{smooth}}$, II_y is proportional to II_x as elements of $S^2T^*_xX\otimes N_xX = S^2T^*_yX\otimes N_yX$.*

Proof. By our hypothesis and Proposition 3.12.1, there exist functions C_s such that $r^\mu_{sik} = C_s q^\mu_{ik}$ for all s, μ, i, k. Under a change of frame $e_s \mapsto e_s + g^0_s e_0$ we have $r^\mu_{sik} \mapsto r^\mu_{sik} + g^0_s q^\mu_{ik}$, and thus we may restrict to frames where $r^\mu_{sik} = 0$. This restriction gives us a well-defined splitting $\hat{F} = \{e_0\} \oplus W$, where $W = \{e_s\}$. But $r^\mu_{sik} = 0$ implies $\omega^j_s = 0$, as for each j there exists a μ with $q^\mu_{j\alpha}\omega^\alpha_0 \neq 0$. We now have

$$de_s \equiv \omega^0_s e_0 \operatorname{mod} W.$$

To complete the proof in this direction we need to show that $\omega^0_s = 0$, as then $\mathbb{P}W$ is a fixed linear space in each tangent space. Consider

$$0 = d\omega^j_s = -\omega^j_0 \wedge \omega^0_s.$$

Since there is more than one j-index by hypothesis, we conclude $\omega^0_s = 0$. Thus $X = J(\mathbb{P}W, Y)$ where Y is any subvariety of X transverse to the fibers of X. □

Exercise 3.12.3: Prove the other direction for Proposition 3.12.2.

Remark 3.12.4. It would be nice to characterize the varieties $X = J(\mathbb{P}W, Y)$ where the Gauss map of Y is not necessarily non-degenerate. A first guess would be that if there is a linear subspace $\hat{W} \subset \hat{T}_{[e_0]}F$ such that II_y is proportional to $II_{[e_0]}$ for all $y \in \mathbb{P}\hat{W} \cap X_{\mathsf{smooth}}$, then there is a splitting $\hat{W} = \{e_0\} \oplus W$ and X is a cone with vertex $\mathbb{P}W$, i.e., $X = J(\mathbb{P}W, Y)$ for some variety Y. But additional hypotheses are necessary in this case, as pointed out in [**130**].

Characterization of the case where $\dim \gamma(X) = 1$.

Theorem 3.12.5. *If $f = n - 1$, then X is a cone over an osculating variety to a curve.*

Exercise 3.12.6: Prove the theorem. Find a curve, as we did in Theorem 3.4.6 for the $n = 2$ case, and recover X from this curve.

Further invariants when $\delta_*(X) = f$. To simplify the calculations in what follows, we will now assume there exists a quadric $q \in |II|$ with $\operatorname{rank}(q) = n - f$, i.e., that the dual defect of X equals f. As mentioned in Remark 3.12.4, without this assumption the problem is considerably more difficult.

Our assumption implies that we may choose q^{n+1} to have rank $n - f$, and we further adapt bases so that $q^{n+1}_{jk} = \delta_{jk}$. (Note that under our assumption q^{n+1} is a generic quadric, so the forms ω^μ_ν are still independent and independent of the semi-basic forms.) Equation (3.17) implies

$$\omega^j_s = r^{n+1}_{sjk}\omega^k_0.$$

Furthermore, assume e_1 is a generic vector in T_xF and, by using the remaining $SO(n-f)$ freedom in the changes of bases among the e_j's, we diagonalize r^{n+1}_{1jk}. Since e_1 is a generic vector, the number of distinct eigenvalues of (r^{n+1}_{1jk}) is an invariant.

Discussion of the "generic" case. We now consider the generic case, where all the eigenvalues of (r^{n+1}_{1jk}) are distinct. Write $r^{n+1}_{1jk} = \lambda_j \delta_{jk}$ and consider

$$\begin{aligned} 0 = d\omega^{n+2}_1 &= \omega^{n+2}_j \wedge \omega^j_1 \\ &= q^{n+2}_{jk} \omega^k_0 \wedge (\lambda_j \omega^j_0) \\ &= \sum_{j<k} q^{n+2}_{jk} (\lambda_j - \lambda_k) \omega^k_0 \wedge \omega^j_0. \end{aligned}$$

We obtain $q^{n+2}_{jk} = 0$ for $j \neq k$ and similarly $q^{\phi}_{jk} = 0$ for all $n+2 \leq \phi \leq n+a$. This has the geometric interpretation that a generic line in F intersects the focal hypersurface $\Phi \subset F$ in $n-f$ distinct points. Recall that the degree of Φ is $n-f$, so our situation means that in the defining equation of Φ (which may be a product of polynomials of lower degree) no factor in the product is repeated. In the language of algebraic geometry, Φ is *reduced* (although not necessarily irreducible).

Theorem 3.12.7. *Let $X^n \subset \mathbb{P}V$ be a subvariety with degenerate Gauss map having $f < n-1$ dimensional fibers. Let F be a general fiber. If the focal hypersurface Φ is reduced, then the first normal space is of dimension at most $n-f$. Moreover, either X is a hypersurface in a $\mathbb{P}^{n+1} \subseteq \mathbb{P}V$, or $II_{X,x}$ contains at least one quadric of rank one.*

Proof. Our calculation above shows that all quadrics in $|II_{X,x}|$ can be simultaneously diagonalized; thus the dimension of the system of quadrics is at most $n-f$. A short calculation shows that the only way the prolongation of such a system can be nonempty is for there to be a quadric of rank one in the system. □

Theorem 3.12.7 overlaps a result in [4], where additional hypotheses are made to insure X is a hypersurface. The essential idea for Theorem 3.12.7 is due to Akivis and Goldberg. For further results, see [**4, 130**]. In particular, a three-dimensional variety with Gauss map of rank two is either a join of two curves, or the union of asymptotic or conjugate lines to a surface, or obtained from the following construction: Take two curves that are simultaneously parametrized on an open subset, say $c(t), e(t)$. Then take the union of planes spanned by the line $\overline{c(t)e(t)}$ and the tangent line to $c(t)$.

3.13. Secant and tangential varieties

Over the next few sections we will outline proofs of the following results of Zak [**157**]:

Theorem 3.13.1 (Zak's theorem on linear normality). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, not contained in a hyperplane, such that $\sigma(X) \neq \mathbb{P}^{n+a}$. Then $a \geq \frac{n}{2} + 2$.*

Theorem 3.13.2 (Zak's theorem on Severi varieties). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety, not contained in a hyperplane, such that $\sigma(X) \neq \mathbb{P}^{n+a}$, with $a = \frac{n}{2} + 2$. Then X is one of the four Severi varieties $\mathbb{AP}^2 \subset \mathbb{P}\mathcal{H}$.*

As a first step, in this section we show how to compute $\dim \sigma(X)$ and $\dim \tau(X)$ from the differential invariants of X at a general point. We then state the Fulton-Hansen Theorem relating secant and tangential varieties. Throughout this section we assume $X \subset \mathbb{P}V$ is a smooth variety.

Dimension of $\tau(X)$. Continuing the notation of §3.2, we may consider $\mathcal{F}^1 \to X$ as a bundle over the smooth points of $\tau(X)$ with projection $(e_0, \dots, e_{n+a}) \mapsto [e_1]$ because $[e_1]$ varies over all smooth points of $\tau(X)$. In other words, $\mathcal{F}^1$ sits above the following incidence correspondence:

$$\begin{tikzcd} & \tilde{T}X \arrow[dl] \arrow[dr] & \\ X & & \tau(X). \end{tikzcd}$$

Thus, to calculate $\dim \tau(X)$ we compute the differential of the projection map onto the second factor. We fix index ranges $2 \leq \rho \leq n$. We have

$$de_1 \equiv \omega_1^0 e_0 + \omega_1^\rho e_\rho + \omega_1^\mu e_\mu \operatorname{mod}\{e_1\},$$

so $\dim \tau(X)$ equals the number of linearly independent forms among the $\omega_1^0, \omega_1^\rho, \omega_1^\mu$. We are free to move e_1 towards any of e_0, e_ρ because we are assuming $[e_1]$ is a general point of $\tau(X)$. Thus, the forms $\omega_1^0, \omega_1^\rho$ are all independent and independent of the semi-basic forms. Thus we need to consider the $\omega_1^\mu = q^\mu_{1\beta}\omega_0^\beta$. Fix $v \in T_xX$ and define

$$\begin{aligned} II_v : T_xX &\to N_xX, \\ w &\mapsto II_{X,x}(v, w). \end{aligned}$$

We conclude:

Proposition 3.13.3 ([**69**]). *Let $x \in X$ be a general point and let $v \in T_xX$ be a generic tangent vector. Then $\dim \tau(X) = n + \operatorname{rank} II_v$.*

This agrees with our expectation that, generally, $\tau(X)$ has dimension $\min\{2n, n + a\}$. If this expectation fails, then we say $\tau(X)$ is *degenerate.*

For X with $\tau(X)$ degenerate, define the *tangential defect* to be

$$\delta_\tau(X) = 2n - \dim \tau(X).$$

Exercise 3.13.4: Calculate the tangential defects of the following varieties:
(a) a curve $X^1 \subset \mathbb{P}^{a+1}$ that is not a line;
(b) the Veronese $v_d(\mathbb{P}^n)$;
(c) the Segre $\mathrm{Seg}(\mathbb{P}^a \times \mathbb{P}^b)$;
(d) the Grassmannian $G(2,6)$.

Dimension of $\sigma(X)$. Recall from Exercise 3.3.12.1 that if $X^n \subset \mathbb{P}^{n+a}$ is a variety, the expected dimension of $\sigma(X)$ is $\min\{2n+1, n+1\}$. If $\sigma(X)$ is not a hypersurface, define

$$\delta_\sigma(X) = 2n + 1 - \dim \sigma(X),$$

the *secant defect* of X, and we say $\sigma(X)$ is *degenerate* if $\delta_\sigma(X) > 0$. While Terracini's Lemma 3.3.11 enables us to easily compute $\dim \sigma(X)$ using one derivative at two general points of X, $\dim \sigma(X)$ is actually subtle to compute using differential invariants at one general point of X. We begin our calculation by first observing that for varieties with degenerate secant varieties, part of the cubic form descends to be a well-defined differential invariant.

For a vector space T and a system of quadrics $A \subset S^2T^*$, recall that

$$\mathrm{Singloc}(A) = \{v \in T \mid v \lrcorner q = 0, \ \forall q \in A\}.$$

For $v \in T = T_xX$, we let

$$\mathrm{Ann}(v) = \mathrm{Ann}_A(v) = \{q \in A \mid v \lrcorner q = 0\}.$$

Following [69], for varieties $X^n \subset \mathbb{P}^{n+a}$ with degenerate tangential varieties, or varieties where $a > n$, one can define a refinement of III at general points $x \in X$. Namely, fixing a II-generic vector $v \in T$, the quantity

$$\omega^\nu_\mu \omega^\mu_\beta \omega^\beta_0 |_{\pi^*(S^3(\mathrm{Singloc}(\mathrm{Ann}(v))))} \otimes (e_\nu \bmod \{\hat{T}, \widehat{II_v(T)}\}) \otimes e^0 \tag{3.18}$$

descends to be well-defined over X because by our hypotheses on the second fundamental form, the relevant variability in F_3 is forced to be zero.

Exercise 3.13.5: Prove (3.18) does indeed descend. ◎

In what follows, we often use the abbreviations $T = T_xX$, $N = N_xX$ and suppress reference to X and the general point $x \in X$.

Define the *refined third fundamental form*

$$III^v \in S^3(\mathrm{Singloc}(\mathrm{Ann}_{|II|}(v)))^* \otimes N/II_v(T)$$

to be given by(3.18).

Exercises 3.13.6:

1. Show that III may be recovered from the forms III^v for all $v \in T$. ⊚
2. Show that if $v \in T$ is a II-generic vector, then $\ker II_v \subset \operatorname{Singloc}(\operatorname{Ann}(v))$.

Points of $\sigma(X)$ are of the form $[e_0 + sf_0]$, where $[e_0], [f_0]$ are points of X and $s \in \mathbb{C}$. Label the corresponding frames for X above $[e_0], [f_0]$ with the same letters. Terracini's Lemma 3.3.11 implies that the dimension of $\sigma(X)$ is one less than the number of linearly independent vectors among $e_0, e_\alpha, f_0, f_\beta$.

Now we use the assumption that X is smooth and connected. We let $v = \underline{e}_1 \in T_{[e_0]}X$. Let $(f_0(t), \dots, f_{n+a}(t))$ be a coframing over an arc $f_0(t)$ such that $(f_0(0), \dots, f_{n+a}(0)) = (e_0, \dots, e_{n+a})$ and $f_0(0)' = e_1$. Expanding $f_0(t), f_\alpha(t)$ in Taylor series, we have

$$\begin{aligned} f_0(t) &= e_0 + te_1 + \frac{t^2}{2} f_0''(0) + \frac{t^3}{3!} f_0'''(0) + O(t^4), \\ f_\alpha(t) &= e_\alpha + t f_\alpha'(0) + \frac{t^2}{2} f_\alpha''(0) + \frac{t^3}{3!} f_\alpha'''(0) + O(t^4). \end{aligned}$$

Now (ignoring twists),

$$\left.\begin{aligned} f_0''(0) &\equiv II(v,v) \\ f_\alpha'(0) &\equiv II(v,\underline{e}_\alpha) \end{aligned}\right\} \bmod \hat{T}_{[e_0]}X$$

and

$$\left.\begin{aligned} f_0'''(0) &\equiv III^v(v,v,v) \\ f_\alpha''(0) &\equiv III^v(v,v,\underline{e}_\alpha \end{aligned}\right\} \bmod \{\hat{T}_{[e_0]}X, II_v(T)\}.$$

Higher-order terms in the series cannot contribute to the dimension of $\sigma(X)$, because either the lower-order terms (together with $\hat{T}_{[e_0]}X$) span a space of maximal dimension and $\sigma(X)$ is nondegenerate, or III^v is identically zero and since the higher fundamental forms lie in the prolongation of the lower fundamental forms, they are zero and the higher-order terms cannot contribute any new directions. So, letting $v \in T$ be II-generic, we obtain the following formula, which was derived in [**69**]:

$$\dim \sigma(X) = \begin{cases} n + \dim II_v(T) & \text{if } III^v(v,v,v) = 0, \\ n + \dim II_v(T) + 1 & \text{if } III^v(v,v,v) \neq 0. \end{cases} \tag{3.19}$$

The Fulton-Hansen Theorem and consequences. Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Notice that by definition, $\tau(X) \subseteq \sigma(X)$. Moreover, the expected dimension of $\tau(X)$ is $\min\{2n, n+a\}$, and that of $\sigma(X)$ is $\min\{2n+1, n+a\}$. The following theorem is proved using an important topological fact about algebraic varieties, namely a version of Zariski's main theorem valid for open varieties (see [**51**] for the proof). It is an application of the celebrated Fulton-Hansen connectedness theorem, which can be found in the same paper, [**51**].

Theorem 3.13.7. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety. Then either*

i. $\dim \tau(X) = 2n$ *and* $\dim \sigma(X) = 2n+1$,

or

ii. $\tau(X) = \sigma(X)$.

Theorem 3.13.7, together with (3.19), implies

Proposition 3.13.8 ([**101**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety such that $\dim \sigma(X) < 2n+1$. Then for all II-generic vectors $v \in T$, we have $III^v = 0$ (and therefore $III_X = 0$). Conversely, if $X^n \subset \mathbb{P}^{n+a}$ is a smooth variety and III_X is not identically zero, then $\sigma(X)$ is nondegenerate.*

Corollary 3.13.9. *The homogeneous varieties $\mathrm{Seg}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_k})$ for $k > 2$, $G(k, k+l)$ for $k, l > 2$, and $v_d(\mathbb{P}^n)$ for $d > 2$ all have nondegenerate secant varieties.*

We will use the Fulton-Hansen Theorem when proving the rank restriction theorems in §3.14, and again when giving a proof of Zak's Theorem on Severi varieties using moving frame techniques in §3.15.

The tangential variety can be generalized as follows: if $Y \subseteq X \subseteq \mathbb{P}^{n+a}$, and $y \in Y$, define $T^\star_y(Y, X)$ to be the union of $\mathbb{P}^1_*$'s, where $\mathbb{P}^1_*$ is a limit of $\mathbb{P}^1_{xy}$'s when $x \in X$ and $y \in Y$, and $x, y \to y_0 \in Y$. Define $\tau(Y, X) = \bigcup_{y \in Y} T^\star_y(Y, X)$, the variety of *relative tangent stars.* As observed by Zak, the Fulton-Hansen Theorem 3.13.7 generalizes to relative tangent stars and joins as follows:

Theorem 3.13.10 ([**157**]). *Let $X^n, Y^y \subset \mathbb{P}V$ be varieties, respectively of dimensions n, y. Assume $Y \subseteq X$. Then either*

i. $\dim J(Y, X) = n + y + 1$ *and* $\dim \tau(Y, X) = n + y$,

or

ii. $J(Y, X) = \tau(Y, X)$.

Using 3.13.10, one easily obtains the famous

Theorem 3.13.11 (Zak's theorem on tangencies [**157**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety not contained in a hyperplane. Let $\dim X_{\mathsf{sing}} = b$, with the convention that $b = -1$ if X is smooth. Let L be any $\mathbb{P}^{n+k} \subset \mathbb{P}^{n+a}$; then $\dim\{x \in X \mid \tilde{T}_x X \subseteq L\} \leq k + (b+1)$.*

Proof. We give the proof assuming X is smooth. Let $Y = \{x \in X \mid \tilde{T}_x X \subseteq L\}$ and let $y = \dim Y$. We have $\tau(Y, X) \subset L$, but since X is not contained in a hyperplane, $\sigma(Y, X) \not\subseteq L$. Thus by 3.13.7, $\dim \tau(Y, X) = y + n$. Thus $y + n \leq n + k$, i.e., $y \leq k$. ☐

Recall that a map $f : X \to Y$ is *finite* if for all $y \in f(X)$, $f^{-1}(y)$ is a finite set.

Corollary 3.13.12. *If X is a smooth variety that is not a linear space, then the Gauss map of X is a finite map.*

3.14. Rank restriction theorems

We already saw that for smooth varieties of small codimension, there are genericity requirements on the projective second fundamental form at general points, namely Theorem 3.10.12. We interpret Theorem 3.10.12 as saying that the second-order infinitesimal geometry (at a general point) can "see" some of the global geometry. Or, to put it another way

In order for X to be smooth, it must "bend enough."

In this section we discuss more general rank restriction theorems, giving a slightly modified version of the rank restriction theorems in [**100**] and [**101**].

We use the convention that if $b = \dim X_{\mathsf{sing}}$, then $b = -1$ if X is smooth.

Theorem 3.14.1 ([**100**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with $b = \dim X_{\mathsf{sing}}$. Let $x \in X$ be a general point and let $\Xi_x \subset \mathbb{P}N^*_x X$ be a subvariety. Let $h \in \Xi_x$ be a general point and let $r = \operatorname{rank} II(h)$. Then*

$$r \geq n + \dim \Xi_x - 2(a-1) - (b+1).$$

Proof. Since $x \in X$ is a general point, we may extend Ξ_x to a neighborhood U of x to give a local fibration $\Xi \to U$ over an open subset $U \subset X$ with the property that, as systems of quadrics, the integer-valued invariants of Ξ_y are the same as those of Ξ_x for all $y \in U$. Let $\mathcal{I}_\Xi$ denote the incidence correspondence with projection maps π_Ξ, ρ_Ξ and let $Z = \rho_\Xi(\Xi) \subset \mathbb{P}V^*$:

$$\begin{array}{c} \mathcal{I}_\Xi = \{(y,H) \mid \tilde{T}_y X \subset H, H \in \Xi_y\}. \\ \swarrow \quad \searrow \\ X \qquad\quad Z \end{array}$$

Observe that

$$\begin{aligned} \dim \pi_\Xi({\rho_\Xi}^{-1}(H)) &= \dim {\rho_\Xi}^{-1}(H), \\ \dim \mathcal{I}_\Xi &= \dim X + \dim \Xi_x, \\ \dim Z = \operatorname{rank} d\rho_{\Xi,(x,H)} &= \dim \mathcal{I}_\Xi - \dim {\rho_\Xi}^{-1}(H). \end{aligned}$$

Putting these together, we conclude that

$$\operatorname{rank} d\rho_{\Xi,(x,H)} = \dim X + \dim \Xi_x - \dim \pi_\Xi {\rho_\Xi}^{-1}(H).$$

Let $\mathcal{F}_\Xi \subset \mathcal{F}^*$ denote the subbundle where $[e^{n+a}] \in \Xi$. On $\mathcal{F}_\Xi$, the $\omega^{n+a}_{n+\lambda}$ are no longer necessarily linearly independent or linearly independent of the semi-basic forms. Using Zak's theorem on tangencies 3.13.11, with $k = a-1$, and observing that $\dim \rho^{-1}(H) \geq \dim {\rho_\Xi}^{-1}(H)$, we obtain

$$\dim\{\omega^{n+a}_{n+\lambda}, \omega^{n+a}_\alpha\} \geq n + \dim \Xi_x - ((a-1)+(b+1)). \tag{3.20}$$

Note that $\dim\{\omega^{n+a}_\alpha\} = r$. At worst we have $\dim\{\omega^{n+a}_{n+\lambda}\} = a-1$, which yields the theorem $\square$

Taking $\Xi_x = \mathbb{P}N^*_x X$, we recover Theorem 3.10.11. Taking Ξ_x to be a point, we obtain

Corollary 3.14.2 ([**100**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with $b = \dim X_{\mathsf{sing}}$. Let $x \in X$ be a general point and let $h \in N^*_x X$. Then*

$$\operatorname{rank} II(h) \geq n - 2(a-1) - (b+1).$$

Corollary 3.14.3 ([**100**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety with $b = \dim X_{\mathsf{sing}}$. Let $x \in X$ be a general point and assume $a < \frac{n-(b+1)}{2} + 1$. Then $III_{X,x} = 0$.*

Exercise 3.14.4: Prove Corollary 3.14.3. ⊚

We will now refine the rank restriction theorem in the case $\sigma(X) \neq \mathbb{P}^{n+a}$. For simplicity, assume $\sigma(X)$ is a hypersurface. (See [**101**] for the general case.) In this case we may smoothly project X to $\mathbb{P}^{n+a-1}$ from a point not on $\sigma(X)$ to obtain a variety of codimension $a-1$, which gives us an immediate improvement of the rank restriction theorem, replacing $a-1$ by $a-2$ in the inequality (3.20).

We get a further improvement as follows: take Ξ_x to be the set of $h \in N^*_x X$ such that $(q^h)_{\mathsf{sing}}$ contains a generic vector. By our assumptions, $\Xi_x \subset \mathbb{P}N^*_x X$ is a hypersurface.

Exercise 3.14.5: Show that Ξ_x is indeed a hypersurface.

We obtain

$$r \geq n - \dim\{\omega^{n+a}_{n+\lambda} \bmod \omega^{n+a}_\alpha\},$$

so at least $r \geq n-(a-1)$. In §3.15 we will define a subbundle of $\mathcal{F}^1$ on which $\dim\{\omega^{n+a}_{n+\lambda} \bmod \omega^{n+a}_\alpha\} \leq a-2$, which will prove

Theorem 3.14.6 (Rank restriction [**101**]). *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate tangential variety that is a hypersurface. Let $x \in X$ be a general point and let*

$$\Xi_x = \mathbb{P}\{h \in N^*_x X \mid II(h)_{\mathsf{sing}} \text{ contains a } II\text{-generic vector}\}.$$

Let r be the rank of a generic quadric in Ξ_x. Then $r \geq n-a+2$.

3.15. Local study of smooth varieties with degenerate tangential varieties

In this section we prove the assertion necessary to prove Theorem 3.14.6, and prove that any smooth variety with a critical secant defect resembles a Severi variety to order two.

Fix a general point $x \in X$ of a smooth variety $X^n \subset \mathbb{P}^{n+a}$ with degenerate tangential variety that is a hypersurface.

Fix a II-generic vector $v \in T$. By Exercise 3.13.6.2 we have the following flag in T:

$$\ker II_v \subset \{v, \ker II_v\} \subseteq \mathrm{Singloc}(\mathrm{Ann}(v)) \subset T. \tag{3.21}$$

Recall that $\dim(\ker II_v) = \delta_\tau(X)$ and $\dim(\mathrm{Singloc}(\mathrm{Ann}(v)) = n - r$, where r is the rank of a generic quadric in $|II|$. We use the following index ranges for the rest of this section: $2 \leq s, t \leq \mathrm{rank}\, II_v - r$, $\mathrm{rank}\, II_v - r + 1 \leq i, j \leq \mathrm{rank}\, II_v$, $\mathrm{rank}\, II_v + 1 \leq \epsilon, \delta \leq n$ and $1 \leq \lambda \leq \mathrm{rank}\, II_v + 1$. For future reference, we record here the way these spaces will be referred to in indices:

$$\begin{aligned} v &= \underline{e}_1, \\ \ker II_v &= \{\underline{e}_\epsilon\}, \\ \mathrm{Singloc}(\mathrm{Ann}(v)) &= \{\underline{e}_1, \underline{e}_\epsilon, \underline{e}_s\} = \{\underline{e}_\xi\}, \\ T/\ker II_v &= \{\underline{e}_1, \underline{e}_j, \underline{e}_s\} = \{\underline{e}_\lambda\}, \\ T &= \{\underline{e}_1, \underline{e}_s, \underline{e}_j, \underline{e}_\epsilon\} = \{\underline{e}_\alpha\}. \end{aligned}$$

Now let $H = [e^{n+a}]$ be the unique hyperplane such that $v \lrcorner\, q^{n+a} = 0$. The vectors $II(\underline{e}_1, \underline{e}_1), II(\underline{e}_1, \underline{e}_s), II(\underline{e}_1, \underline{e}_j)$ form a basis of $II_v(T) \subset N_xX$. Let $\mathcal{F}^{s'} \to X$ denote the frame bundle on which the tangent bundle of X is adapted to the flag (3.21) and furthermore $\underline{e}_{n+\lambda} = II(\underline{e}_1, \underline{e}_\lambda)$. Our adaptations have the effect that

$$\omega_1^{n+a} = 0,\ \omega_1^{n+1} = \omega_0^1,\ \omega_1^{n+j} = \omega_0^j,\ \omega_1^{n+s} = \omega_0^s.$$

We may use motions in the fiber of $\mathcal{F}^{s'}$ corresponding to shearing by g^j_{n+1} and g^j_{n+s} to obtain

$$\begin{aligned} r^{n+a}_{11j} &= 0, \\ r^{n+a}_{1sj} &= 0, \end{aligned} \tag{3.22}$$

and denote the subbundle of $\mathcal{F}^{s\prime}$ where these equations hold by $\mathcal{F}^s$. Now consider the equations for the cubic form (3.5) restricted to $\mathcal{F}^s$:

$$
\begin{aligned}
r^{n+a}_{11\beta}\omega^\beta_0 &= -\omega^{n+a}_{n+1},\\
r^{n+a}_{1j\beta}\omega^\beta_0 &= -\omega^{n+a}_{n+j} + q^{n+a}_{jk}\omega^k_1,\\
r^{n+a}_{1s\beta}\omega^\beta_0 &= -\omega^{n+a}_{n+s}.
\end{aligned} \tag{3.23}
$$

Our assumption that $III^v = 0$ implies

$$
r^{n+a}_{\xi\eta\zeta} = 0, \quad \forall \xi, \eta, \zeta, \tag{3.24}
$$

and (3.24), together with (3.22), implies

$$
\begin{aligned}
\omega^{n+a}_{n+1} &= 0,\\
\omega^{n+a}_{n+s} &= 0,\\
\omega^{n+a}_{n+j} &= q^{n+a}_{jk}\omega^k_1 - r^{n+a}_{1jk}\omega^k_0.
\end{aligned} \tag{3.25}
$$

This implies

$$
\dim\{\omega^{n+a}_{n+\lambda}\} = \dim\{\omega^{n+a}_{n+1}, \omega^{n+a}_{n+j}, \omega^{n+a}_{n+s}\} \leq a - 2,
$$

proving Theorem 3.14.6.

We have

$$
n = \dim(\ker II_v) + \operatorname{rank}(\operatorname{Ann}(v)) + 1 + S = \delta_\tau + r + 1 + S,
$$

where $S = \dim \operatorname{Singloc}(\operatorname{Ann}(v)) - (1 + \dim\ker II_v)$. Combined with the rank restriction theorem 3.14.6, which gives $r \geq n - a + 2$, we have

$$
a \geq \frac{n}{2} + 2 + \frac{S}{2}.
$$

This implies $a \geq \frac{n}{2} + 2$, proving Zak's Theorem 3.13.1 on linear normality.

Varieties with critical tangential defects. We say a smooth variety $X^n \subset \mathbb{P}^{n+a}$ has a critical tangential (or secant) defect if $a = \frac{n}{2} + 2$ and $\sigma(X)$ is degenerate.

We now restrict our study to critical tangential defects. This implies $\tau(X)$ is a hypersurface and $S = 0$, so there are no e_s's.

We restrict to the subbundle of $\mathcal{F}^s$ where $q^{n+a}_{ij} = \delta_{ij}$.

Consider the following coefficients of F_3:

$$
\begin{aligned}
r^{n+a}_{k\epsilon\beta}\omega^\beta_0 &= -q^{n+j}_{k\epsilon}\omega^{n+a}_{n+j} + \omega^k_\epsilon,\\
r^{n+j}_{1\epsilon\beta}\omega^\beta_0 &= \omega^j_\epsilon + q^{n+j}_{\epsilon k}\omega^k_1.
\end{aligned}
$$

Using (3.25), we obtain

$$\omega^k_\epsilon \equiv \sum_j q^{n+j}_{k\epsilon}\omega^j_1 \operatorname{mod}\{\omega^\alpha_0\},$$

$$\omega^k_\epsilon \equiv -q^{n+k}_{j\epsilon}\omega^j_1 \operatorname{mod}\{\omega^\alpha_0\},$$

which imply

$$q^{n+j}_{k\epsilon} + q^{n+k}_{j\epsilon} = 0 \quad \forall j, k, \epsilon.$$

Phrased invariantly, we have shown:

Lemma 3.15.1. *Let $II \in S^2T^*{\otimes}N$ be a critical tangential defect. Fix a generic $v \in T$ to obtain a quadratic form on $T/\operatorname{Singloc}(\operatorname{Ann}(v))$ which is well-defined up to scale, and fix its scale. Then the mapping*

$$\ker II_v \to \operatorname{End}(T/\operatorname{Singloc}(\operatorname{Ann}(v)))$$

takes image in $\mathfrak{so}(T/\operatorname{Singloc}(\operatorname{Ann}(v)))$.

The quadratic form of the lemma is $q^{n+a}\,|_{T/\operatorname{Singloc}(\operatorname{Ann}(v))}$.

Observe that $q^{n+1}|_{\ker II_v}$ is well-defined because

$$\ker II_v \subseteq \operatorname{Baseloc}\{q^{n+2}, \dots, q^{n+a}\}.$$

Consider

$$r^{n+j}_{11\beta}\omega^\beta_0 = -\omega^{n+j}_{n+1} + 2\omega^j_1,$$

$$r^{n+j}_{1\epsilon\beta}\omega^\beta_0 = \omega^j_\epsilon + q^{n+j}_{\epsilon k}\omega^k_1,$$

which imply

$$\omega^{n+j}_{n+1} \equiv 2\omega^j_1 \operatorname{mod}\{\omega^\alpha_0\},$$

$$\omega^j_\epsilon \equiv -q^{n+j}_{\epsilon k}\omega^k_1 \operatorname{mod}\{\omega^\alpha_0\}.$$

Computing

$$r^{n+j}_{\epsilon\delta\beta}\omega^\beta_0 = -q^{n+1}_{\epsilon\delta}\omega^{n+j}_{n+1} + q^{n+j}_{\epsilon k}\omega^k_\delta + q^{n+j}_{\delta k}\omega^k_\epsilon,$$

modding out by the semi-basic forms and using (3.15), we obtain

$$q^{n+j}_{\epsilon k}q^{n+k}_{\delta i} + q^{n+j}_{\delta k}q^{n+k}_{\epsilon i} = -2q^{n+1}_{\epsilon\delta}\delta^i_j \quad \forall \epsilon, \delta, j, k, i. \tag{3.26}$$

Consider the map

$$\ker II_v \to \operatorname{End}(T/q_{\mathsf{sing}}),$$

$$w^\epsilon e_\epsilon \mapsto w^\epsilon q^{n+\kappa}_{j\epsilon}(e_j)^*{\otimes}e_k.$$

The relation (3.26) implies the fundamental lemma A.7.5 of Clifford algebras applies, thus:

Lemma 3.15.2. *Let $II \in S^2T^*{\otimes}N$ have a critical tangential defect. Then $T/\operatorname{Singloc}(\operatorname{Ann}(v))$ is a $Cl(\ker II_v, Q)$-module.*

Keeping in mind that $\dim \ker II_v = a-3$ and $\dim T/\operatorname{Singloc}(\operatorname{Ann}(v)) = a-2$, Exercise A.7.10.4 implies

Corollary 3.15.3. *Let $X^n \subset \mathbb{P}^{n+a}$ be a smooth variety with degenerate secant variety with a critical defect, so $a = \frac{n}{2}+2$. Then the only possibilities are* $\dim(T/\operatorname{Singloc}(\operatorname{Ann}(v))) = a - 2 = 1, 2, 4, 8$.

Examples of these representations have models $\mathbb{A} \oplus \mathbb{A}$; see [**74**].

We now outline the proof that the second fundamental form must be $|II| = \{u\overline{u}, v\overline{v}, u\overline{v}\}$, i.e., the fundamental form of a Severi variety.

We normalize $q^{n+k}_{j\epsilon}$ to correspond to the standard multiplication by J_ϵ, where we take the e_ϵ as standard basis elements of $\mathsf{Im}(\mathbb{A})$. We obtain $q^{n+k}_{ij} = 0$ by applying $\operatorname{Singloc}(\operatorname{Ann}(\underline{e}_k)) \subseteq \operatorname{Baseloc}(\operatorname{Ann}(\underline{e}_k))$. By using a fiber motion, we may normalize $q^{n+1}_{j\beta} = 0$ and $q^{n+1}_{\xi\eta} = \delta_{\xi\eta}$. This puts the system in the standard form of a Severi variety. For more details, see [**101**].

To finish the proof of Zak's theorem on Severi varieties, we need to show that any variety with second fundamental form at a general point isomorphic to the second fundamental form of a Severi variety, and with $III^v = 0$, must be a Severi variety. The Veronese case is treated in Exercise 3.9.9. For the other three cases something stronger is true. The general statement is

Theorem 3.15.4 ([**109**]). *Let $X^n \subset \mathbb{CP}^{n+a}$ be a complex submanifold. Let $x \in X$ be a general point. If $|II| \simeq |II_{Z,z}|$, where Z is a compact rank two Hermitian symmetric space (other than a quadric hypersurface) in its natural embedding, then $\overline{X} = Z$.*

The general technique of proof is to decompose $S^3T^* \otimes N$ into irreducible R-modules, where $R \subset GL(T) \times GL(N)$ is the subgroup preserving $II \in S^2T^* \otimes N$, and to use the higher-order Bertini theorems. For details, see [**109**].

Remark 3.15.5. Zak's original proof of his theorem on Severi varieties involved a detailed study on a potential Severi variety X of quadric sections, i.e., linear sections $X \cap L$ such that $X \cap L \subset L$ is a quadric hypersurface in L. Recently a new proof of Zak's theorem by Chaput [**32**] uses elementary properties of these quadric sections to show immediately that any Severi variety must be homogeneous. Chaput's proof is by far the most elegant to date.

3.16. Generalized Monge systems

We present a generalization of Fubini's Theorem 3.9.1 to characterize smooth intersections of quadrics in small codimension.

Before addressing the question of determining if X is contained in quadric hypersurfaces, let's try an easier one: How many derivatives does one need

to take to determine if X is contained in a hyperplane H? If X is a hypersurface, then two derivatives at a general point are enough by Exercise 3.2.12.2 (if X does not leave its tangent plane to first order at a general point, it must be equal to its tangent plane). On the other hand, if the codimension of X is not fixed, there is no fixed number of derivatives that would guarantee that X is contained in a hyperplane (there are osculating spaces of higher and higher order as one raises the codimension). The following corollary of the rank restriction theorem implies that if $\operatorname{codim}(X)$ is small, the answer is the same as if X were a hypersurface:

Theorem 3.16.1 ([**100**]). *Let $X^n \subset \mathbb{CP}^{n+a}$ be a variety with $a < \frac{n-(b+1)}{2} + 1$ (where $b = \dim X_{\mathsf{sing}}$). Let $x \in X$ be a general point. If a hyperplane H osculates to order two at x, then $X \subset H$.*

Exercise 3.16.2: Prove the theorem. ⊚

Thus in small codimension, two derivatives are sufficient to determine if X is contained in a hyperplane.

In Exercise 1.7.3.2 we saw the classical Monge equation, a fifth-order ODE characterizing plane conics. In §3.9 we saw that in higher dimensions, quadric hypersurfaces are characterized by a third-order PDE (namely $F_3 = l \circ II$). Here we derive a *generalized Monge system* characterizing $I_2(X)$, the set of quadric hypersurfaces containing X, for varieties X such that at general points $III_X = 0$ and II has no linear syzygies (see §3.7).

A variety X is locally the intersection of quadrics if $N^*_x X$ is spanned by the differentials of quadratic equations. Let (x^0, x^α, x^μ) be local linear coordinates on $\mathbb{P}V$ adapted to the point $x = (0,0,0) \in X$, such that $T_x X = \{\frac{\partial}{\partial x^\alpha}\}$. Let (e_0, e_α, e_μ) denote the dual basis. (Before, we denoted the x^A by e^A.)

A basis of S^2V^* is given by $((x^0)^2, x^\alpha x^0, x^\mu x^0, x^\alpha x^\beta, x^\alpha x^\mu, x^\mu x^\nu)$. A basis of the subspace of quadrics in S^2V^* whose zero locus contains $x = [e_0]$ is given by $(x^\alpha x^0, x^\mu x^0, x^\alpha x^\beta, x^\alpha x^\mu, x^\mu x^\nu)$. A basis of the subspace of quadrics in S^2V^* whose zero locus contains $x = [e_0]$ and tangent to X at x (i.e., osculating to order one at x) is given by $(x^\mu x^0, x^\alpha x^\beta, x^\alpha x^\mu, x^\mu x^\nu)$. A basis of the subspace of quadrics in S^2V^* osculating to order two at x, i.e., a basis of the kernel of $\mathbb{FF}^2_{v_2(x),v_2(X)}$, is $(x^\mu x^0 - q^\mu_{\alpha\beta} x^\alpha x^\beta, x^\alpha x^\mu, x^\mu x^\nu)$.

In order that N^*_x be spanned by differentials of quadratic polynomials, it is necessary that

$$\{dP_x | P \in \ker \mathbb{FF}^k_{v_2(X)}\} = N^*_x X \tag{3.27}$$

for all k. (We occasionally suppress reference to X and the basepoint x in what follows.) For $k \leq 2$, (3.27) automatically holds; for $k = 3$ it will hold

if and only if

$$r^\mu_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma} a^\mu_{\nu\gamma} q^\nu_{\alpha\beta} \tag{3.28}$$

for some constants $a^\mu_{\nu\gamma} \in \mathbb{C}$. Notice that if $r^\mu_{\alpha\beta\gamma} = \mathfrak{S}_{\alpha\beta\gamma} a^\mu_{\nu\gamma} q^\nu_{\alpha\beta}$ in some frame, it holds in any choice of frame (with different constants $a^\mu_{\nu\gamma}$), so the expression (3.28) has intrinsic meaning. If (3.28) holds, then

$$\ker \mathbb{FF}^3_{v_2(X)} = \{x^\mu x^0 - q^\mu_{\alpha\beta} x^\alpha x^\beta - a^\mu_{\nu\beta} x^\nu x^\beta, x^\mu x^\nu\}.$$

Continuing in the same fashion, we uncover the following conditions:

$$\begin{aligned} F^\mu_3 &= 3a^\mu_{\nu\gamma}\omega^\gamma_0 II^\nu, \\ F^\mu_4 &= 4a^\mu_{\nu\alpha}\omega^\alpha_0 F^\nu_3 + 3b^\mu_{\nu\tau} II^\nu II^\tau, \\ F^\mu_5 &= 5a^\mu_{\nu\gamma}\omega^\gamma_0 F^\nu_4 + 10 b^\mu_{\nu\tau} F^\nu_3 II^\tau, \end{aligned} \tag{3.29}$$

where $a^\mu_{\nu\alpha}, b^\mu_{\nu\tau} = b^\mu_{\tau\nu} \in \mathbb{C}$. Moreover, if there are no linear syzygies among the quadrics in $|II|$, as explained in §3.7, then $F_6 = 0$; thus, N^*_x is spanned by the differentials of quadrics, and these quadrics are smooth along X, so they generate $I(X)$. In this case, we will call (3.29) the *generalized Monge system for quadrics.*

In summary:

Theorem 3.16.3 ([**102**]). *Let $X \subset \mathbb{P}V = \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Assume $III_{X_x} = 0$ and that there are no linear syzygies in $|II|_x$. Then*

$$\begin{aligned} &\dim\{\textit{quadrics osculating to order three at } x\} \leq a + \binom{a+1}{2} - 1, \\ &\dim\{\textit{quadrics osculating to order four at } x\} \leq a - 1. \end{aligned} \tag{3.30}$$

If the generalized Monge system (3.29) holds, then

$$I_2(X) = \ker \mathbb{FF}^4_{v_2(X)x}.$$

Equality occurs in the first (respectively second) line of (3.30) if and only if the first (resp. second) line of (3.29) holds at x, in which case $I(X)$ is generated by $I_2(X)$. If the generalized Monge system does not hold, then $I(X)$ is not generated by quadrics.

If one assumes appropriate genericity conditions, there exist analogous Monge equations for $I_d(X)$ of order $2d+1$ in small codimension; see [**102**].

3.17. Complete intersections

The least pathological algebraic varieties are the smooth hypersurfaces. For example, the dimension is obvious and the degree is simply the degree of the single polynomial defining X. A class of varieties that share many of the simple properties of hypersurfaces is the class of *complete intersections.*

Definition 3.17.1. A variety $X^n \subset \mathbb{P}^{n+a}$ is a *complete intersection* if the ideal of X, $I(X)$, can be generated by a elements.

Context from algebraic geometry. The classical Lefschetz theorem on hyperplane sections [**68**] states that for a smooth complete intersection, most of the topology is inherited from the ambient projective space. Barth and Barth-Larsen proved theorems partially generalizing Lefschetz's results to arbitrary smooth varieties of small codimension [**8**]. This led Hartshorne to raise the question as to whether all smooth varieties of small codimension must be complete intersections, namely his famous conjecture:

Conjecture 3.17.2 (Hartshorne's conjecture on complete intersections, [**73**]). *Let $X^n \subset \mathbb{CP}^{n+a}$ be a smooth variety. If $a < \frac{n}{2}$, then X is a complete intersection*

This conjecture has inspired an extraordinary amount of work (see [**126, 114, 131, 46**]), although it is as open today as the day Hartshorne made it. Zak's theorem on linear normality (which Hartshorne conjectured at the same time) can be viewed as a first-order approximation to Hartshorne's conjecture; see [**73**].

So far, all our studies have been of questions that have been primarily local in nature. Even when we dealt with global properties (e.g., smoothness of a variety), we transformed the problem to a local study (e.g., infinitesimal study of the dual variety). Hartshorne's conjecture is a fundamentally global problem, yet nevertheless we will still work locally.

Singular hypersurfaces. Singular hypersurfaces are the key to understanding the projective geometry of non-complete intersections. To explain why, for simplicity, assume for the moment that X is the intersection of hypersurfaces of degree d.

Proposition 3.17.3 ([**102**]). *Let $X \subset \mathbb{P}V$ be a variety such that $I(X)$ is generated by $I_d(X)$ and $I_{d-1}(X) = (0)$. Then the following are equivalent:*

1. X is a complete intersection.

2. Every hypersurface of degree d containing X is smooth at all $x \in X_{\mathsf{smooth}}$.

3. Let $x \in X_{\mathsf{smooth}}$. Every hypersurface of degree d containing X is smooth at x.

Proof. Fix $x \in X_{\mathsf{smooth}}$. We have the surjective map

$$I_d(X) \to N_x^*X,$$
$$P \mapsto dP_x.$$

Since $\dim N_x^*X = a$, X is a complete intersection iff the map is injective. $\square$

Thus if X is a complete intersection, any hypersurface that is singular at any $x \in X_{\mathsf{smooth}}$ cannot contain X. If X is not a hypersurface, there are always singular hypersurfaces in $I(X)$. The proposition says that the singularities occur away from X.

For the general case, we use terminology due to Lvovsky [**116**]:

Definition 3.17.4. Let $X \subset \mathbb{P}V$ be a variety. Let $P \in I_d(X)$ and let $Z = Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. We will say Z *trivially contains* X if $P = l^1 P_1 + \dots l^m P_m$ with $P_1, \dots, P_m \in I_{d-1}(X)$ and $l^1, \dots, l^m \in V^*$, and otherwise that Z *essentially contains* X.

Proposition 3.17.5 (A local characterization of complete intersections [**102**]). *Let $X \subset \mathbb{P}V$ be a variety. The following are equivalent:*

1. X is a complete intersection.

2. Every hypersurface essentially containing X is smooth at all $x \in X_{\mathsf{smooth}}$.

3. Let $x \in X_{\mathsf{smooth}}$. Every hypersurface essentially containing X is smooth at x.

Exercise 3.17.6: Prove the proposition.

Proposition 3.17.5 localizes the study of complete intersections to a point, and furthermore filters the conormal bundle at that point to enable us to study one degree at a time. However, to determine if a hypersurface essentially contains X, one might need to take an arbitrarily high number of derivatives. To have computable conditions, one could work with osculating hypersurfaces rather than the hypersurfaces containing X. The advantage is that one would only need to study a fixed number of derivatives for each fixed degree of hypersurface; the disadvantage is that one will only obtain sufficient conditions to be a complete intersection. By the results in §3.7, at best one could prove that there are no singular hypersurfaces of degree d osculating to order $2d+2$ at x, and that the first restrictions one could hope for are at order $d+1$.

We now specialize to the case $d = 2$:

Intersections of quadrics. Looking at (3.14), we see that $\ker \mathbb{FF}^3_{v_2(X),x}$ is as small as possible if there are no linear syzygies among the quadrics in $II_{X,x}$.

We have the following lemma relating linear syzygies and ranks of quadrics:

Lemma 3.17.7 ([**102**]). *Let $A^p \subset S^2T^*$ be a p-dimensional system of quadrics on an n-dimensional vector space. Say there is a linear syzygy*

$$l^1 Q_1 + \dots + l^p Q_p = 0,$$

where both $l^i \in T^*$ *and* $Q_i \in A$ *are independent sets of vectors. Then* $\operatorname{rank} Q \leq 2p$, $\forall Q \in A$.

If one now compares Lemma 3.17.7 with the rank restriction theorem, one sees that if $a < \frac{n-(b+1)}{3}$ then there are no linear syzygies in $|II|$. Combined with the generalized Monge system, we obtain

Theorem 3.17.8 ([**102**]). *Let* $X^n \subset \mathbb{P}^{n+a}$ *be a variety and* $x \in X$ *a general point. Let* $b = \dim X_{\mathsf{sing}}$. *(Set* $b = -1$ *if* X *is smooth.) If* $a < \frac{n-(b+1)}{3}$, *then*

$$\begin{aligned} &\dim\{\textit{quadrics osculating to order three at } x\} \leq a + \binom{a+1}{2} - 1, \\ &\dim\{\textit{quadrics osculating to order four at } x\} \leq a - 1. \end{aligned} \tag{3.31}$$

Equality occurs in the first (respectively second) line of (3.31) *if and only if the generalized Monge system holds to order three (respectively four) at* x. *If the generalized Monge system holds, then* X *is a complete intersection of the* $(a-1)$*-dimensional family of quadrics osculating to order four.*

Corollary 3.17.9 ([**102**]). *Let* $X^n \subset \mathbb{P}^{n+a}$ *be a variety, let* $x \in X$ *be a general point and let* $b = \dim X_{\mathsf{sing}}$. *If* $a < \frac{n-(b+1)}{3}$, *then any quadric osculating to order four at* x *is smooth at* x *and any quadric osculating to order five at* x *contains* X.

Corollary 3.17.10 ([**102**]). *Let* $X^n \subset \mathbb{P}^{n+a}$ *be a variety whose ideal is generated in degree two, and let* $b = \dim X_{\mathsf{sing}}$. *If* $a < \frac{n-(b+1)}{3}$, *then* X *is a complete intersection.*

There are other results concluding that a variety must be a complete intersection, but most of them we are aware of only apply in codimension two (see, e.g., [**46**]). As with Corollary 3.17.9, all such results impose some additional hypotheses, so cannot be taken as evidence for or against Hartshorne's conjecture, but just indicating where to look for a potential counter-example.

Chapter 4

Cartan-Kähler I: Linear Algebra and Constant-Coefficient Homogeneous Systems

We have seen that differentiating the forms that generate an exterior differential system often reveals additional conditions that integral manifolds must satisfy (e.g., the Gauss and Codazzi equations for a surface in Euclidean space). The conditions are consequences of the fact that mixed partials must commute. What we did not see was a way of telling when one has differentiated enough to find all hidden conditions. We do know the answer in two cases: If a system is in Cauchy-Kowalevski form there are no extra conditions. In the case of the Frobenius Theorem, if the system passes a first-order test, then there are no extra conditions.

What will emerge over the next few chapters is a test, called *Cartan's Test*, that will tell us when we have differentiated enough.

The general version of Cartan's Test is described in Chapter 7. For a given integral element $E \in \mathcal{V}_n(\mathcal{I})_x$ of an exterior differential system $\mathcal{I}$ on a manifold Σ, it guarantees existence of an integral manifold to the system with tangent plane E if E passes the test.

In Chapter 5, we present a version of Cartan's Test valid for a class of exterior differential systems with independence condition called *linear Pfaffian systems*. These are systems that are generated by 1-forms and have the additional property that the variety of integral elements through a

point $x \in \Sigma$ is an affine space. The class of linear Pfaffian systems includes all systems of PDE expressed as exterior differential systems on jet spaces. One way in which a linear Pfaffian system is simpler than a general EDS is that an integral element $E \in \mathcal{V}_n(\mathcal{I}, \Omega)_x$ passes Cartan's Test iff all integral elements at x do.

In this chapter we study first-order, constant-coefficient, homogeneous systems of PDE for analytic maps $f : V \to W$ expressed in terms of *tableaux*. We derive *Cartan's Test* for this class of systems, which determines if the initial data one might naïvely hope to specify (based on counting equations) actually determines a solution.

We dedicate an entire chapter to such a restrictive class of EDS because at each point of a manifold Σ with a linear Pfaffian system there is a naturally defined tableau, and the system passes Cartan's Test for linear Pfaffian systems at a point $x \in \Sigma$ if and only if its associated tableau does and the *torsion* of the system (defined in Chapter 5) vanishes at x.

In analogy with the inverse function theorem, Cartan's Test for linear Pfaffian systems (and even in its most general form) implies that if the linear algebra at the infinitesimal level works out right, the rest follows. What we do in this chapter is determine what it takes to get the linear algebra to work out right.

Throughout this chapter, V is an n-dimensional vector space, and W is an s-dimensional vector space. We use the index ranges $1 \leq i, j, k \leq n$, $1 \leq a, b, c \leq s$. V has the basis $v_1, \ldots, v_n$ and V^* the corresponding dual basis $v^1, \ldots, v^n$; W has basis $w_1, \ldots, w_s$ and W^* the dual basis $w^1, \ldots, w^s$.

4.1. Tableaux

Let $x = x^i v_i, u = u^a w_a$ denote elements of V and W respectively. We will consider $(x^1, \ldots, x^n)$, respectively $(u^1, \ldots, u^n)$, as coordinate functions on V and W respectively. Any first-order, constant-coefficient, homogeneous system of PDE for maps $f : V \to W$ is given in coordinates by equations

$$B_a^{ri} \frac{\partial u^a}{\partial x^i} = 0, \qquad 1 \leq r \leq R, \tag{4.1}$$

where the B_a^{ri} are constants. For example, the Cauchy-Riemann system $u^1_{x^1} - u^2_{x^2} = 0, u^1_{x^2} + u^2_{x^1} = 0$ has $B^{11}_1 = 1$, $B^{12}_2 = -1$, $B^{12}_1 = 0$, $B^{11}_2 = 0$, $B^{21}_2 = 1$, $B^{22}_1 = 1$, $B^{21}_1 = 0$ and $B^{22}_2 = 0$.

To phrase (4.1) in a coordinate-free manner, given a map $f : V \to W$, we define a "Gauss map"

$$\gamma_f : V \to V^* \otimes W = \mathrm{Hom}(V, W),$$
$$x \mapsto \frac{\partial f^a(x)}{\partial x^i} w_a \otimes v^i,$$

where we identify $T_xV \simeq V, T_{f(x)}W \simeq W$ and translate the differential to the origin. In other words, we identify $J^1_x(V, W)/J^0_x(V, W)$ with $W \otimes V^*$. More generally, we will identify $J^k_x(V, W)/J^{k-1}_x(V, W)$ with $W \otimes S^kV^*$, the homogeneous W-valued polynomials of degree k on V.

Our system may be described free of coordinates as a subspace $B \subset V \otimes W^*$, where

$$B = \{B^{ri}_a v_i \otimes w^a | 1 \leq r \leq R\}.$$

We think of B as the space of equations. A map f is a solution if B annihilates $\gamma_f(x)$ for all $x \in V$, i.e.,

$$\langle b, \gamma_f(x)\rangle = 0 \quad \forall x \in V, b \in B.$$

B is often called the space of *symbol relations*. (We will see that it generalizes the *principal symbol* used in standard PDE terminology.) We think of $B^\perp \subset W \otimes V^*$ as the space of admissible first derivatives (see below). In computations we will often use $B^\perp$ rather than B, so it has a special name:

Definition 4.1.1. A *tableau* is a linear subspace $A \subseteq W \otimes V^*$. A tableau A determines a first-order, constant-coefficient, homogeneous system of PDE for maps $f : V \to W$, namely the system whose solutions are those maps f satisfying $\gamma_f(V) \subseteq A$.

Note that systems defined by a tableau always have solutions

$$f(x) = f_0 + A_0 x,$$

where $f_0 \in W$ and $A_0 \in A$. We will be interested in what higher-order terms can appear in the Taylor series of a solution.

Example 4.1.2. The equation $u^1_{x^1} + u^2_{x^2} = 0$ has symbol relations

$$B = \{w^1 \otimes v_1 + w^2 \otimes v_2\} \subset W^* \otimes V$$

and tableau

$$A = B^\perp = \{w_1 \otimes v^1 - w_2 \otimes v^2, w_1 \otimes v^2, w_2 \otimes v^1\} \subset W \otimes V^*. \tag{4.2}$$

Often it will be convenient to identify V with $\mathbb{R}^n$, W with $\mathbb{R}^s$, and to use our fixed basis to write our expressions in matrix form. For example, (4.2) becomes

$$A = \left\{ \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

Example 4.1.3. The Cauchy-Riemann system $u^1_{x^1} = u^2_{x^2}$, $u^1_{x^2} = -u^2_{x^1}$ has tableau

$$\begin{aligned} A =& \{a(w_1 \otimes v^1 + w_2 \otimes v^2) + b(-w_2 \otimes v^1 + w_1 \otimes v_2) | a, b \in \mathbb{R}\} \\ \simeq& \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R} \right\}. \end{aligned}$$

Example 4.1.4. The tableau $A = (0)$ corresponds to a Frobenius system. The equations are $u^a_{x^i} = 0$, $\forall i, a$. The only solutions to this system are constant maps. We will say *solutions depend on s constants.*

Example 4.1.5. When $A = W \otimes V^*$, there are no equations and any map is a solution. Here, *solutions depend on s functions of n variables.*

Example 4.1.6. Let $L^* \subset V^*$ be a k-dimensional linear subspace and let $A = W \otimes L^*$. If $L^* = \{v^1, \dots, v^k\}$, the equations are $u^a_{x^\rho} = 0$, $k+1 \leq \rho \leq n$, and the solutions are $u^a(x^1, \dots, x^n) = f^a(x^1, \dots, x^k)$, where the f^a are arbitrary functions. Here, *solutions depend on s functions of k variables.* With respect to adapted bases we may write A in block form as $(*, 0)$, where the left block is free and the right block is zero.

Example 4.1.7. Let $A = Y \otimes V^*$, where Y is a p-dimensional linear subspace of W. If Y is spanned by u^λ, $1 \leq \lambda \leq p$, the equations are $u^\xi_{x^i} = 0$, where $p + 1 \leq \xi \leq s$. Solutions are

$$\begin{aligned} u^\lambda(x^1, \dots, x^n) &= f^\lambda(x^1, \dots, x^n), \\ u^\xi(x^1, \dots, x^n) &= f^\xi_0, \end{aligned}$$

where the $f^\lambda(x^1, \dots, x^n)$ are arbitrary functions and the f^ξ_0 are constants; so, *solutions depend on p functions of n variables and $s - p$ constants.* One may think of this tableau as a combination of two disjoint systems of types 4.1.5 and 4.1.6. With respect to adapted bases we may write A in block form as $\binom{*}{0}$, where the top block is free and the bottom block is zero.

In the rest of this chapter, we will develop a test for a tableau such that if it passes, we will know exactly what initial data one should specify to determine an analytic solution to (4.1). What to do if the tableau fails the test is taken up in Chapter 5.

Given a tableau A, suppose we try to specify the solution in terms of a power series

$$u^a(x) = p^a + p^a_i x^i + p^a_{ij} x^i x^j + p^a_{ijk} x^i x^j x^k + \dots. \tag{4.3}$$

Since the system is homogeneous with constant coefficients, u is a solution if and only if each term satisfies the system and the series converges. As we have seen, there is no restriction on the constant terms, and the coefficients

of the linear term must satisfy $p^a_i w_a \otimes v^i \in A$. Moving on to the coefficients of the quadratic term in the series, the quadratic term must be such that $\gamma_{p^a_{ij}x^ix^j}(x) \in A, \quad \forall x \in V$. Thus,

$$p^a_{ij}x^i w_a \otimes v^j \in A \ \forall x \in V,$$

so that

$$(p^a_{ij}w_a \otimes v^j) \otimes v^i \in (A \otimes V^*).$$

Since mixed partials commute, $(p^a_{ij}w_a \otimes v^i) \otimes v^j$ is also an element of $W \otimes S^2V^*$; in indices this means $p^a_{ij} = p^a_{ji}$. Combining these two conditions gives

$$p^a_{ij}w_a \otimes v^i \otimes v^j \in (A \otimes V^*) \cap (W \otimes S^2V^*) =: A^{(1)},$$

where, at the risk of being redundant, the first factor is due to the equations and the second to the commuting of derivatives. The space $A^{(1)}$ is called the *(first) prolongation* of A.

Similarly, the condition on the next term of (4.3) is

$$p^a_{ijk}w_a \otimes v^i \otimes v^j \otimes v^k \in (A \otimes V^* \otimes V^*) \cap (W \otimes S^3V^*) =: A^{(2)}, \tag{4.4}$$

and so on for all l. In general, we define

$$A^{(l)} := (A \otimes V^{*\otimes l}) \cap (W \otimes S^{(l+1)}V^*)$$

to be the l^{th} *prolongation* of A. A map $f : V \to W$, given in terms of a convergent Taylor series, is a solution to the system defined by A if and only if for all k, the tensor formed from the k-th term lies in $A^{(k-1)}$.

Exercises 4.1.8:

1. Show that $A^{(l)}$ is naturally identified with the set of solutions to the system defined by A that are homogeneous polynomials of degree $l+1$.
2. Show that if $A^{(l)} = 0$ for some l, then solutions depend on a finite number of constants.

Definition 4.1.9. A *tableau of order p* is a linear subspace $A \subset W \otimes S^pV^*$. It determines a homogeneous constant-coefficient system of PDE of order p for W-valued functions on V. In particular, the $(p-1)$-st prolongation of a tableau (of order one) is a tableau of order p. For a tableau A of order p, we define its prolongations by $A^{(l)} = (A \otimes V^{*\otimes l}) \cap (W \otimes S^{p+l}V^*)$.

Example 4.1.10. The equation $u_{xy} = 0$ may be encoded as a tableau of order two as

$$A = \{p_{11}v^1 \circ v^1 + p_{22}v^2 \circ v^2 \mid p_{11}, p_{22} \in \mathbb{R}\}.$$

Here $V = \mathbb{R}^2, W = \mathbb{R}$.

Exercises 4.1.11:

1. Express the system $u_{xx} + v_{yy} = 0$ of one equation for two functions of two variables as a tableau of order two. What is its dimension? ◎
2. Show that $A^{(k+l)} = (A^{(k)})^{(l)}$.

We would like to determing the space of solutions to (4.1) by performing a finite calculation; in particular, we would like to avoid calculating $A^{(l)}$ for all l. As it turns out, we will calculate $A^{(1)}$, and if the system passes a certain test, we will then know the dimension of $A^{(l)}$ for all l and the 'size' of the space of local solutions. To develop some intuition for what the test should be, we examine some examples.

4.2. First example

Consider the tableau $A = (p_1, 0, \ldots, 0)$, where p_1 is a column vector whose entries are free. (This is Example 4.1.6 in the case $\dim L = 1$.) This tableau corresponds to equations $\frac{\partial u^a}{\partial x^\rho} = 0$, where $2 \leq \rho \leq n$. Its solutions are $u^a(x^1, \ldots, x^n) = f^a(x^1)$. In particular, specifying the values $u^a(x^1, 0, \ldots, 0)$ uniquely determines a solution, and all solutions are obtained this way.

We generalize to the tableau

$$\begin{aligned} A &= \{(p_1^a v^1 + C^a_{2b} p_1^b v^2 + \ldots + C^a_{nb} p_1^b v^n) \otimes w_a | p_1^a \in \mathbb{R},\ 1 \leq a \leq s\} \\ &= \{(p_1, C_2 p_1, \ldots, C_n p_1)\}, \end{aligned} \tag{4.5}$$

where the C_ρ are fixed $s \times s$ matrices and $p_1 = (p_1^a)$ is any column vector. The symbol relations corresponding to (4.5) are

$$B = \{w^a \otimes v_\rho - C^a_{\rho b} w^b \otimes v_1 | 2 \leq \rho \leq n, 1 \leq a \leq s\},$$

and the corresponding differential equations are

$$\frac{\partial u^a}{\partial x^\rho} - C^a_{\rho b} \frac{\partial u^b}{\partial x^1} = 0, \qquad \rho = 2, \ldots, n. \tag{4.6}$$

We saw that when $C_\rho \equiv 0 \,\forall \rho$, any convergent series with coefficients $p_1^a, p_{11}^a, p_{111}^a, \ldots$ determines a solution u. The hope is that under some conditions the solutions of the system (4.6) are describable in a similar way. In fact, we will determine when solutions to (4.6) can be given in terms of s arbitrary functions of one variable, and will see that this is the largest space of solutions one could hope for.

Assume we are given constants $p_1^a, p_{11}^a, p_{111}^a, \cdots$. Then we may determine the remaining p_I^a for any multi-index $I = (i_1, \ldots, i_k)$ as follows: To have $(p_i^a w_a \otimes v^i) \in A$, the terms p_ρ^a must be given by $p_\rho^a = C^a_{\rho b} p_1^b$. To have $(p_{ij}^a w_a \otimes v^i v^j) \in A^{(1)}$, the other terms are determined by the p_{11}^a:

$$\begin{aligned} p^a_{\rho 1} &= C^a_{\rho b} p^b_{11}, \\ p^a_{\rho\sigma} &= C^a_{\rho b} p^b_{\sigma 1} = C^a_{\rho b} C^b_{\sigma c} p^c_{11}. \end{aligned} \tag{4.7}$$

These equations imply $\dim A^{(1)} \leq s$. They may lead to conflicting equations because it is also necessary that $p^a_{\rho\sigma} = p^a_{\sigma\rho}$, i.e., that $C^a_{\rho b} C^b_{\sigma c} p^c_{11} = C^b_{\sigma c} C^a_{\rho b} p^c_{11}$.

In other words, to ensure that **no** choice of p^a_{11}'s leads to a conflict, it is necessary that

$$(4.8) \qquad [C_\rho, C_\sigma]^a_c = C^a_{\rho b} C^b_{\sigma c} - C^a_{\sigma b} C^b_{\rho c} = 0 \quad \forall\, a, c, \rho, \sigma.$$

If (4.8) holds, we are free to specify not only p^a_{11} but in fact any convergent power series for $u^a(x^1, 0, \ldots, 0)$. More precisely,

Proposition 4.2.1. *If the C_ρ in tableau (4.5) are such that $[C_\rho, C_\sigma] = 0\ \forall \rho, \sigma$, then there exists a unique solution to the initial value problem*

$$\gamma_u(x) \in A = (p_1, C_2 p_1, \ldots, C_n p_1)$$

with initial condition

$$u^a(x^1, 0, \ldots, 0) = f^a(x^1).$$

Geometrically, the $f^a(x^1)$ determine a curve in $V \times W$, and we would like to enlarge this curve to an n-dimensional integral manifold. By (4.7) we see that there is at most one such, and the proposition states that if (4.8) holds, this candidate is in fact an integral manifold. If (4.8) fails to hold, there may still be solutions to (4.5), but there will be additional conditions on the functions f^a.

Proof of 4.2.1. We have seen that in this case the $p^a_{\rho i}$ are exactly determined by the p^a_{11}, and the same computation shows the $p^a_{\rho ij}$ are exactly determined by the p^a_{111}, etc. (The commutation relations take care of all possible conflicts.) It remains to show that the formal power series for $u(x^1, \ldots, x^n)$ determined by these coefficients converges.

Let $c = \max\{1, \text{maximum eigenvalue of any } C_\rho\}$. Consider the system

$$\frac{\partial U^a}{\partial x^\rho} = c \frac{\partial U^a}{\partial x^1}, \qquad \rho = 2, \ldots, n.$$

Its general solution is given by

$$U^a(x^1, \ldots, x^n) = F^a(x^1 + c(x^2 + \ldots + x^n)).$$

where $F^a(t)$ are analytic functions of one variable. If

$$F^a(t) = p^a + p^a_1 t + p^a_{11} t^2 + p^a_{111} t^3 + \ldots,$$

then we may use the functions F^a to construct a formal solution to (4.5):

$$\begin{aligned}(4.9)\quad & u^a(x^1, \ldots, x^n) \\ &= p^a + (p^a_1 x^1 + C^a_{\rho b} p^b_1 x^\rho) + (p^a_{11} (x^1)^2 + C^a_{\rho b} p^b_{11} x^1 x^\rho + C^a_{\rho b} C^b_{\sigma c} p^c_{11} x^\rho x^\sigma) \\ &\quad + (p^a_{111} (x^1)^3 + C^a_{\rho b} p^b_{111} (x^1)^2 x^\rho \\ &\quad + C^a_{\rho b} C^b_{\sigma c} p^c_{111} x^1 x^\rho x^\sigma + C^a_{\rho b} C^b_{\sigma c} C^c_{\tau d} p^d_{111} x^\rho x^\sigma x^\tau) + \ldots\end{aligned}$$

On the other hand, we see that any derivative of u is bounded by the corresponding derivative of U:

$$\left|\frac{\partial^I u^a}{\partial x^{i_1}\dots x^{i_d}}(0)\right| \le \left|\frac{\partial^I U^a}{\partial x^{i_1}\dots x^{i_d}}(0)\right|.$$

Since the series (4.9) for u is bounded in norm by a convergent series, it must converge as well. $\square$

Remark 4.2.2. The proof of 4.2.1 is an example of the method of *majorants*.

In summary, the largest space of solutions one could hope for in a system of the form (4.5) is that solutions depend on s functions of 1 variable, and whether or not this is the case can be determined just by computing $A^{(1)}$. One always has $\dim A^{(1)} \le s$, and solutions depend on s functions of one variable if and only if $\dim A^{(1)} = s$.

4.3. Second example

Now we will study a generalization of the simple tableau

$$A = \begin{pmatrix} p_1^\lambda & p_2^\lambda & 0 \\ p_1^\xi & 0 & 0\end{pmatrix} = \begin{pmatrix} p_1 & p_2 & 0 \\ q_1 & 0 & 0\end{pmatrix}, \quad p_1^\lambda, p_2^\lambda, p_1^\xi \in \mathbb{R}, \tag{4.10}$$
$$1 \le \lambda, \mu \le k, \qquad k+1 \le \xi, \eta \le s.$$

This tableau corresponds to the equations $u^\xi_{x^2} = 0$, $u^a_{x^3} = 0$. Its solutions depend on k functions of 2 variables and $s-k$ functions of 1 variable, namely $u^\xi = f^\xi(x^1)$ and $u^\lambda = f^\lambda(x^1, x^2)$. Note that here $\dim A^{(1)} = s + 2k$, since one is free to specify $p^a_{11}, p^\lambda_{12}, p^\lambda_{22}$, while all the other p^a_{ij} must be zero.

Our generalization is

$$A = \begin{pmatrix} p_1 & p_2 & Fp_1 + Gq_1 + Hp_2 \\ q_1 & Cp_1 + Dq_1 + Ep_2 & Ip_1 + Jq_1 + Kp_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \tag{4.11}$$

where $C, E, I, K \in M_{(s-k)\times k}$, $D, J \in M_{(s-k)\times(s-k)}$, $F, H \in M_{k\times k}$, $G \in M_{k\times(s-k)}$, and $M_{r\times s}$ denotes the set of $r \times s$ matrices. In this tableau there are s independent entries in the first column and k new independent entries in the second column. So, the entries in p_1, p_2, q_1 are all independent and all other entries are linear combinations of them.

The system of differential equations corresponding to A is

$$\begin{aligned}
\frac{\partial u^\xi}{\partial x^2} &= C^\xi_\lambda \frac{\partial u^\lambda}{\partial x^1} + D^\xi_\eta \frac{\partial u^\eta}{\partial x^1} + E^\xi_\lambda \frac{\partial u^\lambda}{\partial x^2}, \\
\frac{\partial u^\xi}{\partial x^3} &= I^\xi_\lambda \frac{\partial u^\lambda}{\partial x^1} + J^\xi_\eta \frac{\partial u^\eta}{\partial x^1} + K^\xi_\lambda \frac{\partial u^\lambda}{\partial x^2}, \\
\frac{\partial u^\mu}{\partial x^3} &= F^\mu_\lambda \frac{\partial u^\lambda}{\partial x^1} + G^\mu_\eta \frac{\partial u^\eta}{\partial x^1} + H^\mu_\lambda \frac{\partial u^\lambda}{\partial x^2}.
\end{aligned}$$

As before, for any solution we are free to specify the order zero data arbitrarily, and first-order data must belong to A. For the second-order terms of a power series solution, we see that we could at most specify the vectors p_{11}, p_{12}, p_{22}, and q_{11}, where $p_{ij} = {}^t(p^1_{ij} \cdots p^k_{ij})$ and $q_{ij} = {}^t(p^{k+1}_{ij} \cdots p^s_{ij})$. Then the remainder of the p^a_{ij} are determined by (4.11), e.g.,

$$\begin{aligned}
q_{21} &= Cp_{11} + Dq_{11} + Ep_{21} = q_{12},\\
q_{22} &= Cp_{12} + Dq_{12} + Ep_{22} = Cp_{12} + D(Cp_{11} + Dq_{11} + Ep_{21}) + Ep_{22},\\
p_{31} &= Fp_{11} + Gq_{11} + Hp_{21}.
\end{aligned}$$

Now there are two ways to describe $q_{32} = q_{23}$, namely

$$\begin{aligned}
q_{32} &= Ip_{12} + Jq_{12} + Kp_{22}\\
&= Ip_{12} + J(Cp_{11} + Dq_{11} + Ep_{21}) + Kp_{22},\\
q_{23} &= Cp_{13} + Dq_{13} + Ep_{23}\\
&= C(Fp_{11} + Gq_{11} + Hp_{21}) + D(Ip_{11} + Jq_{11} + Kp_{21})\\
&\quad + E(Fp_{12} + G(Cp_{11} + Dq_{11} + Ep_{21}) + Hp_{22}).
\end{aligned}$$

In order to have consistent equations for all choices of $p_{11}, p_{12}, p_{22}, q_{11}$, the four matrix equations

$$\begin{aligned}
JC &= CF + DI + EGC,\\
I + JE &= CH + DK + EF + EGE,\\
K &= EH,\\
JD &= CG + DJ + EGD
\end{aligned} \tag{4.12}$$

must hold. (These are obtained by setting the p_{11}, p_{12}, p_{22} and q_{11} coefficients of q_{23} equal to the corresponding coefficients of q_{32}.) It is easy to check that if (4.12) holds, we are able to freely specify $p_{111}, q_{111}, p_{112}, p_{122}, p_{222}$. These exactly determine the value of the 2-jet within $A^{(2)}$; similarly, for $l > 2$ an element of $A^{(l)}$ is determined by $q_{1,\dots,1}$ and $p_{i_1,\dots,i_l}$ with $i_1, \dots, i_l \in \{1, 2\}$.

Proposition 4.3.1. *If the matrix equations* (4.12) *hold, then there is a unique solution to the initial value problem*

$$\gamma_u(x) \subset A,$$

where A is the tableau (4.11) *with initial condition*

$$u^\lambda(x^1, x^2, 0) = f^\lambda(x^1, x^2), \quad u^\xi(x^1, 0, 0) = f^\xi(x^1),$$

where f^λ, f^ξ are analytic.

Proof. The only thing to check is the claim about the determination of $A^{(l)}$ and the convergence of the power series. These are left to the reader. □

For any tableau of the form (4.11), we have $\dim A^{(1)} \leq s + 2k$. Proposition 4.3.1 says that, when equality holds, solutions depend on k functions of 2 variables and $(s-k)$ functions of 1 variable, which is the largest possible space of solutions.

Remark 4.3.2 (*another view of 4.3.1*). One could attempt to solve the system (4.11) by solving a sequence of Cauchy problems. For example, suppose

$$\begin{aligned} u^\xi(x^1,0,0) &= f^\xi(x^1),\\ u^\lambda(x^1,x^2,0) &= f^\lambda(x^1,x^2) \end{aligned}$$

are the functions we specify to determine a putative solution to the system (4.11). We could first solve the Cauchy problem

(4.13)
$$\begin{aligned} &u^\xi_{x^2}(x^1,x^2,0) = C^\xi_\lambda u^\lambda_{x^1}(x^1,x^2,0) + D^\xi_\eta u^\eta_{x^1}(x^1,x^2,0) + E^\xi_\lambda u^\lambda_{x^2}(x^1,x^2,0),\\ &\quad u^\xi(x^1,0,0) = f^\xi(x^1) \end{aligned}$$

to determine $u^\xi(x^1,x^2,0)$, next solve the Cauchy problem

(4.14)
$$\begin{aligned} &u^\lambda_{x^3}(x^1,x^2,x^3) = F^\lambda_\mu u^\mu_{x^1}(x^1,x^2,x^3) + G^\lambda_\xi u^\xi_{x^1}(x^1,x^2,x^3) + E^\lambda_\mu u^\mu_{x^2}(x^1,x^2,x^3),\\ &\quad u^\lambda(x^1,x^2,0) = f^\lambda(x^1,x^2) \end{aligned}$$

to determine $u^\lambda(x^1,x^2,x^3)$, and finally obtain a map $u : V \to W$ by solving

(4.15)
$$\begin{aligned} &u^\xi_{x^3}(x^1,x^2,x^3) = I^\xi_\lambda u^\lambda_{x^1}(x^1,x^2,x^3) + J^\xi_\eta u^\eta_{x^1}(x^1,x^2,x^3) + K^\xi_\lambda u^\lambda_{x^2}(x^1,x^2,x^3),\\ &\quad u^\xi(x^1,x^2,0) = \text{solution to (4.14)}. \end{aligned}$$

Solving (4.13) determines the constants $q_2, q_{12}, q_{22}, q_{112}, \ldots$ for which all subscripts are 1 or 2, with at least one being 2. Solving (4.14) determines $p_3, p_{13}, \ldots$ and solving (4.15) determines $p_{23}, p_{33}, \ldots$.

Just as with the similar procedure we tried in Example 1.2.3, this procedure yields a function which may not solve our system. As with Example 1.2.3, there are other sequences of Cauchy problems—e.g., we could have solved first for $u^\xi(x^1,0,x^3)$—to obtain a function. However, condition (4.12) guarantees that the function we obtain is actually a solution to our system and that no matter which sequence of Cauchy problems we solve, we always obtain the same answer. (Again, the failure of (4.12) to hold does not exclude the possibility that some sequence of Cauchy problems will yield a solution for certain initial data; but (4.12) guarantees that this process will work with all choices of initial data.)

4.4. Third example

To begin with, suppose the tableau A is of the form

$$A = \begin{pmatrix} p_1^1 & p_2^1 & \dots & p_k^1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & p_k^{s_k} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \dots & 0 \\ \vdots & p_2^{s_2} & 0 & \vdots & 0 & \dots & 0 \\ \vdots & 0 & \vdots & \vdots & 0 & \dots & 0 \\ p_1^{s_1} & \vdots & \vdots & \vdots & 0 & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 & \dots & 0 \end{pmatrix}. \tag{4.16}$$

Then $\dim A^{(1)} = s_1 + 2s_2 + 3s_3 + \ldots + ks_k$, since an element of $A^{(1)}$ is determined by a choice of s_1 constants $p_{11}^1, \ldots, p_{11}^{s_1}$, then $2s_2$ constants $p_{12}^1, \ldots, p_{12}^{s_2}$, $p_{22}^1, \ldots, p_{22}^{s_2}$, then $3s_3$ constants $p_{13}^1, \ldots, p_{13}^{s_3}$, $p_{23}^1, \ldots, p_{32}^{s_3}$, $p_{33}^1, \ldots, p_{33}^{s_3}$, and so on, up to ks_k constants $p_{1k}^1, \ldots, p_{kk}^{s_k}$.

The general solution of the corresponding system of differential equations is

$$\begin{aligned} u^a &= f^a(x^1, \ldots, x^k), & 1 &\le a \le s_k, \\ u^a &= f^a(x^1, \ldots, x^{k-1}), & 1 + s_k &\le a \le s_{k-1}, \\ &\vdots \\ u^a &= f^a(x^1), & 1 + s_2 &\le a \le s_1, \\ u^a &= u_0^a \text{ (constant)}, & 1 + s_1 &\le a \le s. \end{aligned}$$

If we generalize tableau (4.16) by inserting linear combinations of the entries occurring above and to the left of each zero, we obtain a tableau of the same dimension, which satisfies the inequality

$$\dim A^{(1)} \le s_1 + 2s_2 + \ldots + ks_k$$

with equality iff any choice of the $s_1 + 2s_2 + \ldots + ks_k$ constants p_{ij}^a for a point in $A^{(1)}$ determines the other p_{ij}^a's without conflict. Moreover, in this case $A^{(k)}$ has its maximum possible dimension for all k as well, i.e., commutation of all second-order derivatives implies commutation of all higher-order derivatives.

4.5. The general case

Definition 4.5.1. Given a tableau $A \subset W \otimes V^*$ expressed in terms of bases $b = (v^1, \dots, v^n)$ of V^* and $q = (w_1, \dots, w_s)$ of W, let $s_1(b), \dots, s_n(b)$ be defined by

$$\begin{aligned} s_1(b) &= \text{\# of independent entries in the first column of } A, \\ s_1(b) + s_2(b) &= \text{\# of independent entries in the first 2 columns of } A, \\ &\vdots \\ s_1(b) + \dots + s_n(b) &= \text{\# of independent entries in } A = \dim A. \end{aligned}$$

In other words, $s_k(b)$ is the number of new independent entries in the k-th column.

Note that the $s_k(b)$ do not depend on the choice of basis for W, but only on the flag $F = (F_0, F_1, \dots, F_n)$ of subspaces in V^* induced by b, which we denote by

$$F_j = \{v_1, \dots, v_j\}^\perp = \{v^{j+1}, \dots, v^n\},$$

with $F_0 = V^*$ and $F_n = (0)$. Hence, we write $s_k(F)$ instead of $s_k(b)$.

We define

$$A_k(F) = (W \otimes F_k) \cap A$$

and observe that

$$\dim A_k(F) = s_{k+1}(F) + \dots + s_n(F). \tag{4.17}$$

One can visualize $A_k(F)$ as the subspace of matrices in A for which the first k columns are zero, when we use the basis b for V. In our first example, $A_1 = (0)$; in our second example, A_1 is the k-dimensional subspace of A obtained by setting $p_1, q_1 = 0$.

Definition 4.5.2. Let $A \subset W \otimes V^*$ be a tableau. Define

$$\begin{aligned} &s_1(A) = \max\{s_1(F) |\ \text{all flags}\}, \\ &s_2(A) = \max\{s_2(F) |\ \text{flags with } s_1(F) = s_1(A)\}, \\ &\quad\vdots \\ &s_n(A) = \max\{s_n(F) |\ \text{flags with } s_1(F) = s_1(A), \dots, s_{n-1}(F) = s_{n-1}(A)\}. \end{aligned}$$

The s_i are called the *(reduced) characters* of A. They are invariants of A with respect to the action of $GL(V) \times GL(W)$. We will call a flag F an *A-generic flag* when $s_i(F) = s_i$ for all i. (We will often write s_i instead of $s_i(A)$ when there is no risk of confusion.)

Notice that $s \geq s_1 \geq \dots \geq s_n \geq 0$. For example, tableau (4.5) has characters $s_1 = s, s_2 = s_3 = \dots s_n = 0$, while (4.10) has characters $s_1 = s$, $s_2 = k$, $s_3 = 0$.

We fix an A-generic flag F induced by a basis b, and will suppress further reference to it. Given $U \subseteq W \otimes S^d V^*$, we define

$$U_k := U \cap (W \otimes S^d \{v^{k+1}, \ldots, v^n\}).$$

Note that

$$(A_k)^{(1)} = (A^{(1)})_k.$$

Proposition 4.5.3.

$$\dim A^{(1)} \leq s_1 + 2s_2 + 3s_3 + \ldots + ns_n. \tag{4.18}$$

Proof. We have the exact sequence

$$0 \to A_k{}^{(1)} \to A_{k-1}{}^{(1)} \to A_{k-1},$$

in which the last map is $p \mapsto v_k \lrcorner p$. Thus

$$\dim A_{k-1}{}^{(1)} - \dim A_k{}^{(1)} \leq \dim A_{k-1}.$$

Summing for $1 \leq k \leq n$ (and recalling that $A_0 = A$), we have

$$\dim A^{(1)} \leq \dim A + \dim A_1 + \ldots + \dim A_{n-1},$$

which, along with (4.17), gives the inequality of the proposition. □

Definition 4.5.4. A tableau $A \subset W \otimes V^*$ is said to be *involutive* if equality holds in (4.18).

For the "model" involutive tableau (4.16), solutions are uniquely determined by specifying

$$u^\sigma(x^1, \ldots, x^k, 0, \ldots, 0) \text{ for } \sigma \leq s_k.$$

This motivates the following:

Definition 4.5.5. For an integer σ between 1 and s, define the *level* of σ to be the largest k such that $\sigma \leq s_k$. (If $\sigma > s_1$, its level is defined to be zero.)

Example 4.5.6. Tableau (4.10) has characters $s_1 = s, s_2 = s - k, s_3 = 0$, so

$$\text{level}(1) = \text{level}(2) = \ldots = \text{level}(s-k) = 2$$

while

$$\text{level}(s-k+1) = \ldots = \text{level}(s) = 1.$$

For any tableau, the most general set of initial data one could hope to specify is

$$u^\sigma(x^1, \ldots, x^k, 0, \ldots, 0), \text{ where level}(\sigma) = k, \tag{4.19}$$

with an integral manifold obtained by then solving a sequence of Cauchy problems. In fact, an argument that is conceptually the same as in the examples of the previous sections yields

Theorem 4.5.7 (Cartan-Kähler for tableaux). *Let $A \subset W \otimes V^*$ be a tableau. Choose A-generic bases of V, W which induce coordinates x^i on V and u^a on W. If A is involutive, then any choice of analytic functions (4.19) uniquely determines an integral manifold of the differential system represented by A in some neighborhood of the origin.*

Definition 4.5.8. If A is an involutive tableau such that $s_l \neq 0$ but $s_{l+1} = 0$, then s_l is called the *character* of the system and the number l is called the *Cartan integer*.

According to Theorem 4.5.7, a solution is determined by specifying

$$\begin{aligned} &s_l \text{ functions of } l \text{ variables,} \\ &s_{l-1} - s_l \text{ functions of } l-1 \text{ variables,} \\ &\vdots \\ &s_1 - s_2 \text{ functions of 1 variable} \\ \text{and } &s_1 - s \text{ constants.} \end{aligned}$$

The freedom of the functions of l variables is more significant than all the other choices in the sense that, for large p, in the pth-order term in the Taylor series expansion of a solution there will be many more coefficients coming from the functions of l variables than the others. Thus we usually will say that for an involutive EDS with character s_l the integral manifolds *depend on s_l functions of l variables*, and ignore the rest.

Exercises 4.5.9:

1. Show that a tableau $A \subset W \otimes V^*$ having characters $s_1 = s_2 = \dots s_{n-1} = s, s_n = 0$ is always involutive.
2. Consider the tableau for the Cauchy-Riemann equations $u_{x_i} = v_{y_i}, u_{y_i} = -v_{x_i}$ for two functions u, v of $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$. Calculate the characters and show that the tableau is involutive. Describe the size of the space of solutions based on Cartan's Test, and say how this coincides with how you learned how to construct solutions of this system in your first complex analysis course.
3. Let $A \subset W \otimes V^*$ be any tableau and consider the enlarged tableau
$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \subset \tilde{W} \otimes \tilde{V}^*,$$
where V, W are subspaces of $\tilde{V}, \tilde{W}$ respectively. Show that A and $\tilde{A}$ have the same nonzero characters and that $\tilde{A}$ is involutive if and only if A is.
4. Let $\dim V = n$ and endow V with a volume form and inner product. Recall from §2.6 the Laplacian $\Delta = *d*d + d*d* : \Omega^p(V) \to \Omega^p(V)$, a second-order differential operator. (Here we identify V with its tangent space at any point.)

(a) The tableau for Laplace's system $\Delta f = 0$ for functions $f : V \to W$ is $W \otimes S^2_0 V^* \subset W \otimes S^2 V^*$, where $S^2_0 V^*$ is the traceless symmetric tensors in $V^* \otimes V^*$. (We give V an inner product and the trace is with respect to that inner product.) What are the symbol relations?
(b) If $\dim V = \dim W = 2$, show that the Cauchy-Riemann system implies Laplace's system in the sense that any solution to the Cauchy-Riemann system must also solve the Laplace system.
(c) Consider a p-form α as a $\Lambda^p V$-valued function on V. Write down the tableau of the system defined by the two equations $d\alpha = 0$ and $d * \alpha = 0$ for $p = 1$, and show that its prolongation is contained in $W \otimes S^2_0 V^*$, where $W = \Lambda^p V$.

4.6. The characteristic variety of a tableau

In Theorem 4.5.7 we do not necessarily obtain all solutions to an involutive system by a choice of analytic functions as specified in the theorem. We only obtain those that can be obtained by specifying *non-characteristic* initial data, that is, initial data based on a choice of A-generic flag for V^*. (The others can be obtained by the Cartan algorithm described in Chapter 5, but it involves specializations to non-generic submanifolds.) If we choose a non-generic flag, the corresponding Cauchy problem may not have any solutions, or it may be undetermined, with an infinite number of solutions.

Example 4.6.1. Consider the system

$$u_x - v_y = 0, \quad u_y - v_x = 0. \tag{4.20}$$

If we pick initial data for u, v along a line through the origin other than $x = \pm y$, then this extends to a unique solution to the system. But if we specify initial data along the line $y = x$, then unless $u - v$ is constant this cannot be extended to a solution. If $u - v$ is constant along $y = x$, then there are an infinite number of extensions to a local solution.

Definition 4.6.2. For $\xi \in V^*$, define the *symbol mapping* at ξ by

$$\sigma_\xi : W \to (W \otimes V^*)/A,$$
$$\sigma_\xi(w) = w \otimes \xi \bmod A.$$

Definition 4.6.3. Define $\Xi_A \subset \mathbb{P}V^*$, the *characteristic variety of A*, to be

$$\Xi_A = \{[\xi] \in \mathbb{P}V^* \mid \ker \sigma_\xi \neq 0\}.$$

We can interpret Ξ_A as the set of hyperplanes in $\mathbb{P}V$ for which the extension of an $(n-1)$-dimensional integral element to an n-dimensional integral element is not unique; see Theorem 5.7.7.

Example (4.6.1 *continued*). The tableau A corresponding to (4.20) consists of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. This has characters $s_1 = 2 = \dim A$ and $s_2 =$

0. A line in V spanned by ${}^t(x, y)$ is characteristic if the system $ax + by = 0$, $bx + ay = 0$ of equations for a, b has rank less than two. Of course, this happens for the two lines $y = \pm x$. Thus Ξ_A consists of the two points $[1, 1]$ and $[1, -1]$.

Example 4.6.4. Suppose $\dim V = 2$. If $s_2 \neq 0$, then $\Xi_A = \mathbb{P}V^*$. If $s_1 = s$ and $s_2 = 0$, then Ξ_A is a collection of points.

Example 4.6.5. Let V and W be three-dimensional, and let A be the five-dimensional space of matrices of the form

$$\begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix}.$$

It is easy to see that A has characters $s_1 = 3, s_2 = 2, s_3 = 0$. We have

$$\sigma_\xi(w) \equiv \begin{pmatrix} \xi_1 w_1 & \xi_2 w_1 & \xi_3 w_1 \\ \xi_1 w_2 & \xi_2 w_2 & \xi_3 w_2 \\ \xi_1 w_3 & \xi_2 w_3 & \xi_3 w_3 \end{pmatrix} \bmod A.$$

In order for this matrix to be zero modulo A, we must have $\xi_2 w_1 = \xi_1 w_2$, $\xi_2 w_3 = \xi_3 w_2$, and $\xi_1 w_3 = \xi_2 w_2 = \xi_3 w_1$ for some nonzero w. The first two equations show we must have $w_1 = \frac{\xi_1}{\xi_2} w_2$ and $w_3 = \frac{\xi_3}{\xi_2} w_2$. Plugging this into the fourth equation, we obtain $\xi_1 \xi_3 = \xi_2^2$ and the last equation is redundant. Thus the characteristic variety is the conic $\{\xi_1 \xi_3 - \xi_2^2 = 0\}$ in $\mathbb{P}V^* \cong \mathbb{RP}^2$.

Exercise 4.6.6: Consider the four-dimensional tableau A of matrices of the form

$$\begin{pmatrix} a & b & c \\ b & a & b \\ d & b & a \end{pmatrix}.$$

Show that Ξ_A consists of four points in $\mathbb{P}V^*$.

Even when we are only interested in real-valued solutions of the underlying PDE, it is useful to consider the *complex characteristic variety*

$$\Xi_A^{\mathbb{C}} := \{[\xi] \in \mathbb{P}V_{\mathbb{C}}^* \mid \ker \sigma_\xi^{\mathbb{C}} \neq 0\},$$

where $\sigma_\xi^{\mathbb{C}} : W_{\mathbb{C}} \to (W_{\mathbb{C}} \otimes V_{\mathbb{C}}^*)/A_{\mathbb{C}}$ is the complexification of σ_ξ (see Appendix C).

Definition 4.6.7. We say A is *determined* if $\dim A^\perp = s$, i.e., the number of equations is the same as the number of unknowns.

Exercise 4.6.8: Show that if A is determined and $\Xi_A^{\mathbb{C}} \neq \mathbb{P}V_{\mathbb{C}}^*$, then A is involutive.

Definition 4.6.9. Recall from Chapter 3 that the *Segre variety*

$$\mathrm{Seg}(\mathbb{P}W \times \mathbb{P}V^*) \subset \mathbb{P}(W \otimes V^*)$$

is the set of decomposable vectors in $\mathbb{P}(W \otimes V^*)$.

Let p_{V^*} be the projection from Seg onto the first factor:

$$\begin{tikzcd} & \mathrm{Seg}(\mathbb{P}V^* \times \mathbb{P}W) \arrow[dl, "p_{V^*}"'] \arrow[dr] & \\ \mathbb{P}V^* & & \mathbb{P}W. \end{tikzcd}$$

Exercise 4.6.10: Let $A \subset W \otimes V^*$ be a tableau. Show that

$$\Xi_A = p_{V^*}(\mathbb{P}A \cap \mathrm{Seg}(\mathbb{P}V^* \times \mathbb{P}W)). \tag{4.21}$$

If $U \subseteq W \otimes S^p V^*$ is a tableau of order p, let $v_p : \mathbb{P}V^* \to \mathbb{P}S^p V^*$ denote the Veronese re-embedding $[v] \mapsto [v^p]$ (see Chapter 3), and define $\Xi_U = p_{V^*}(\mathbb{P}U \cap \mathrm{Seg}(\mathbb{P}W \otimes v_p(\mathbb{P}V^*)))$.

Exercises 4.6.11:

1. If in a given basis the symbol relations are spanned by R matrices B^{ri}_a of dimension $n \times s$, and $\xi = \xi_i v^i$, then σ_ξ is given by the $R \times s$ matrix $(B^{ri}_a \xi_i)$. In particular, if $R < s$, then $\Xi_A = \mathbb{P}V^*$.
2. Again show that if $R < s$ then $\Xi_A = \mathbb{P}V^*$, this time using (4.21) as the definition of Ξ_A, and the fact that varieties of complementary dimension must intersect in projective space.
3. Show that if $U = A^{(k)}$ then $\Xi_U = \Xi_A$.

Dimension and degree of the characteristic variety. In this subsection, we work over $\mathbb{C}$ and drop the $\mathbb{C}$ from the notation.

We estimate the dimension of Ξ_A and a modification of the degree of Ξ_A in terms of the characters of A. We use the convention that the empty set $\emptyset$ has dimension -1, and denote the modified degree, which will be defined below, by $\widetilde{\deg}$.

Our motivation is the following theorem:

Theorem 4.6.12. *Let $A \subset W \otimes V^*$ be a tableau. Non-characteristic integral manifolds of the exterior differential system induced by A depend on $\dim \Xi^{\mathbb{C}}_A + 1$ functions of $\widetilde{\deg}(\Xi^{\mathbb{C}}_A)$ variables.*

We will prove this theorem in Chapter 5. It follows from first proving that some prolongation of A is involutive and the characteristic variety is unchanged under prolongation.

Let A be a tableau with Cartan integer k. First note that $\dim \Xi_A \leq k-1$, because $\mathbb{P}F_k \cap \Xi_A = \emptyset$ if $F_k \subset V$ is a generic codimension k subspace. Thus the projection $\pi : \mathbb{P}V^* \to \mathbb{P}(V^*/F_k)$ restricted to Ξ_A is a finite map. (If it is

surjective, its degree is $\deg(\Xi_A)$.) We need to define $\widetilde{\deg}$ precisely because π need not be finite.

To arrive at the definition of $\widetilde{\deg}$ we begin with the case where A is such that $k = 1$, and hence $\dim(\Xi_A) \leq 0$.

After choosing adapted bases, we may write A as a subspace of the space of $n \times s$ matrices: $A = (p, C_2 p, \dots, C_n p)$, where the C_ρ are $s \times s$ matrices and $p \in \mathbb{R}^{s_1} \subset \mathbb{R}^s$ is variable, ${}^t p = (p_1, \dots, p_{s_1}, 0, \dots, 0)$. Here $\Xi_A \neq \emptyset$ iff there is a choice of p such that the matrix $A(p)$ is of rank one. (If $s_1 = s$ this occurs if and only if there exists a simultaneous eigenvector for the C_ρ). In particular, if A is involutive then $\Xi_A \neq \emptyset$.

How many points can be in Ξ_A? If the C_ρ are all simultaneously diagonalizable in the upper $s_1 \times s_1$ blocks with zeros elsewhere, then there are s_1 points, and since these s_1 points span A, this is the most possible. It may happen that one can normalize to upper $s_1 \times s_1$ blocks with zeros elsewhere but with simultaneous eigenspaces of dimension greater than one in the nonzero block. This lowers the number of points, but for each eigenspace of dimension μ, there is a vector $\xi \in V^*$ such that there is a μ-dimensional subspace $U \subseteq W$ such that $[\xi \otimes w] \in (\mathbb{P}A \cap \mathrm{Seg})$ for all $w \in U$.

Since the condition of primary interest is involutivity, we consider a modification of the degree:

We decompose Ξ_A into irreducible components $\Xi_A = \bigcup_\alpha \Xi_A^\alpha$, where the components Ξ_A^α are of maximal dimension. We set

$$\widetilde{\deg}\Xi_A = \sum_\alpha \mu^\alpha \deg \Xi_A^\alpha,$$

where μ^α is the dimension of ${p_{V^*}}^{-1}([\xi])$ for a general point $[\xi] \in \Xi_A^\alpha$.

We prove the following result, which is a special case of Theorem V.3.6 in [**20**]. The general case is a theorem about the characteristic sheaf of a linear Pfaffian system, and is much more difficult to prove due to the possible presence of nilpotents in the stalk (i.e., the characteristic scheme at a point). In fact the proof in [**20**] uses the Grothendieck-Riemann-Roch theorem.

Theorem 4.6.13. *Let $A \subset W \otimes V^*$ be a tableau with $s_k(A) \neq 0$ and $s_{k+1}(A) = 0$. Then $\dim \Xi_A \leq k - 1$, and equality holds if and only if $\mathbb{P}F_{k-1} \cap \Xi_A \neq \emptyset$, i.e., $\mathbb{P}A_{k-1} \cap \mathrm{Seg}(\mathbb{P}F_{k-1} \times \mathbb{P}W) \neq \emptyset$. In particular, equality holds if A_k is involutive. If $\dim \Xi_A = k - 1$, then $\widetilde{\deg}\Xi_A \leq s_k$, with equality if and only if A_k is involutive.*

Proof. We have already seen the first inequality. Equality occurs iff $\mathbb{P}F_{k-1} \cap \Xi_A \neq \emptyset$, i.e., iff $\mathbb{P}A_{k-1} \cap \mathrm{Seg}(\mathbb{P}F_{k-1} \times \mathbb{P}W) \neq \emptyset$. But as a tableau, A_{k-1} has characters $s_1(A_{k-1}) = s_k(A)$, $s_2(A_{k-1}) = 0$, so we are reduced to the special

case. Since $\mathbb{P}F_{k-1}$ is a generic linear space, the same calculation as above is also valid for the degree. $\square$

The dimension and $\widetilde{\deg}$ of the characteristic variety may be easier to compute than carrying out Cartan's Test. More importantly, this theorem is an important step towards proving Theorem 4.6.12, stated above.

Chapter 5

Cartan-Kähler II: The Cartan Algorithm for Linear Pfaffian Systems

We now generalize the test from Chapter 4 to a test valid for a large class of exterior differential systems called *linear Pfaffian systems*, which are defined in §5.1. In §§5.2–5.4 we present three examples of linear Pfaffian systems that lead us to Cartan's algorithm and the definitions of torsion and prolongation, all of which are given in §5.5. For easy reference, we give a summary and flowchart of the algorithm in §5.6. Additional aspects of the theory, including characteristic hyperplanes, Spencer cohomology and the Goldschmidt version of the Cartan-Kähler Theorem, are given in §5.7. In the remainder of the chapter we give numerous examples, beginning with elementary problems coming mostly from surface theory in §5.8, then an example motivated by variation of Hodge structure in §5.9, then the Cartan-Janet Isometric Immersion Theorem in §5.10, followed by a discussion of isometric embeddings of space forms in §5.11 and concluding with a discussion of calibrations and calibrated submanifolds in §5.12.

5.1. Linear Pfaffian systems

Recall that a Pfaffian system on a manifold Σ is an exterior differential system generated by 1-forms, i.e., $\mathcal{I} = \{\theta^a\}_{\mathsf{diff}}$, $\theta^a \in \Omega^1(\Sigma)$, $1 \leq a \leq s$. If $\Omega = \omega^1 \wedge \ldots \wedge \omega^n$ represents an independence condition, let $J := \{\theta^a, \omega^i\}$

and $I := \{\theta^a\}$. We will often use J to indicate the independence condition in this chapter, and refer to the system as (I, J).

Definition 5.1.1. (I, J) is a *linear Pfaffian system* if $d\theta^a \equiv 0 \bmod J$ for all $1 \leq a \leq s$.

Exercise 5.1.2: Let (I, J) be a linear Pfaffian system as above. Let π^ϵ, $1 \leq \epsilon \leq \dim \Sigma - n - s$, be a collection of 1-forms such that $T^*\Sigma$ is locally spanned by $\theta^a, \omega^i, \pi^\epsilon$. Show that there exist functions $A^a_{\epsilon i}, T^a_{ij}$ defined on Σ such that

$$d\theta^a \equiv A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \bmod I. \tag{5.1}$$

Example 5.1.3. The canonical contact system on $J^2(\mathbb{R}^2, \mathbb{R}^2)$ is a linear Pfaffian system because

$$\begin{aligned} d(du - p^1_1 dx - p^1_2 dy) &= -dp^1_1 \wedge dx - dp^1_2 \wedge dy \\ &\equiv 0 \operatorname{mod}\{dx, dy, du - p^1_1 dx - p^1_2 dy, dv - p^2_1 dx - p^2_2 dy\}, \\ d(dv - p^2_1 dx - p^2_2 dy) &= -dp^2_1 \wedge dx - dp^2_2 \wedge dy \\ &\equiv 0 \operatorname{mod}\{dx, dy, du - p^1_1 dx - p^1_2 dy, dv - p^2_1 dx - p^2_2 dy\}, \end{aligned}$$

and the same calculation shows that the pullback of this system to any submanifold $\Sigma \subset J^2(\mathbb{R}^2, \mathbb{R}^2)$ is linear Pfaffian. More generally, we have

Example 5.1.4. Any system of PDE expressed as the pullback of the contact system on $J^k(M, N)$ to a subset Σ is a linear Pfaffian system. If M has local coordinates $(x^1, \ldots, x^n)$ and N has local coordinates $(u^1, \ldots, u^s)$, then $J^k = J^k(M, N)$ has local coordinates (x^i, u^a, p^a_i, p^a_L), where $L = (l_1, \ldots, l_p)$ is a symmetric multi-index of length $p \leq k-1$. In these coordinates, the contact system I is $\{\theta^a = du^a - p^a_i dx^i, \theta^a_L = dp^a_L - p^a_{Lj} dx^j\}$, and $J = \{\theta^a, \theta^a_L, dx^i\}$. On $J^k(M, N)$,

$$\left.\begin{aligned} d\theta^a &= -dp^a_j \wedge dx^j \\ d\theta^a_L &= -dp^a_{Lj} \wedge dx^j \end{aligned}\right\} \equiv 0 \quad \operatorname{mod} J,$$

and these equations continue to hold when we restrict to any subset $\Sigma \subset J^k$.

Example 5.1.5. On $\mathbb{R}^6$, let $\theta = y^1 dy^2 + y^3 dy^4 + y^5 dx$, let $I = \{\theta\}$ and $J = \{\theta, dx\}$. Then

$$\begin{aligned} d\theta &= dy^1 \wedge dy^2 + dy^3 \wedge dy^4 + dy^5 \wedge dx \\ &\equiv (dy^3 - \frac{y^3}{y^1} dy^1) \wedge dy^4 \operatorname{mod}\{\theta, dx\}. \end{aligned}$$

In this case, (I, J) is not a linear Pfaffian system.

Exercises 5.1.6:
1. On $ASO(n+s)$, with the notation from §2.5, let $I = \{\omega^a\}$ and $J = \{\omega^a, \omega^j\}$. The integral manifolds of (I, J) are the first-order adapted lifts of immersed submanifolds $M^n \subset \mathbb{E}^{n+s}$. Show that (I, J) is a linear Pfaffian system.
2. On $ASO(n+s) \times \mathbb{R}^{s\binom{n+1}{2}}$, let the second factor have coordinates $h^a_{jk} = h^a_{kj}$. Let $I = \{\omega^a, \omega^a_j - h^a_{jk}\omega^k\}$ and $J = \{\omega^a, \omega^a_j - h^a_{jk}\omega^k, \omega^j\}$. Show that the integral manifolds of (I, J) are the first-order adapted lifts of immersed submanifolds $M^n \subset \mathbb{E}^{n+s}$ together with their second fundamental forms, and that (I, J) is a linear Pfaffian system.

In the following three sections, we will arrive at the Cartan algorithm informally, via examples, and then state the algorithm formally in §5.5.

5.2. First example

First, let's reconsider our system of Example 1.2.3:

$$\theta := du - A(x, y, u)dx - B(x, y, u)dy$$
$$\Omega := dx \wedge dy.$$

We differentiated, and obtained

$$d\theta \equiv (A_y - B_x + A_u B - B_u A) dx \wedge dy \mod I,$$

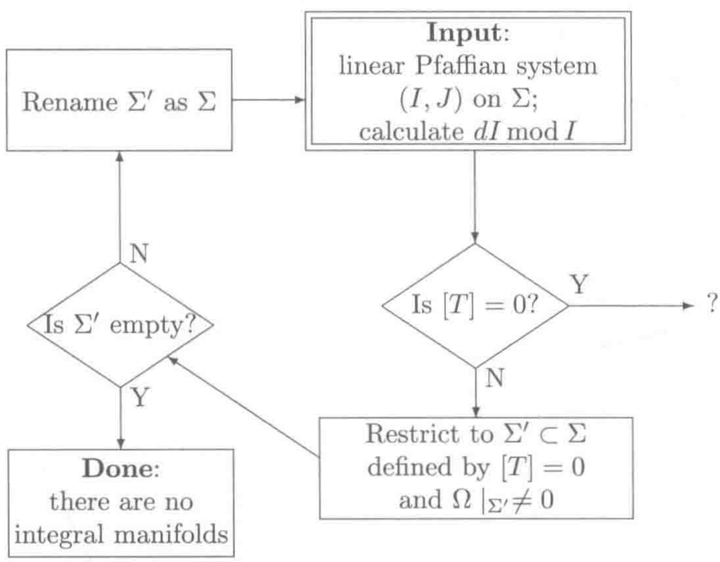
and saw that we had to restrict to the submanifold $\Sigma' \subset \Sigma = \mathbb{R}^3$ defined by $A_y - B_x + A_u B - B_u A = 0$. The term $A_y - B_x + A_u B - B_u A$, which is an obstruction to the existence of integral manifolds, is a special case of *torsion*, which will be defined invariantly in §5.5. For now, we use the following coordinate definition: we say that *apparent torsion* is present if in the expression (5.1) the functions T^a_{ij} are not identically zero. We will say the *apparent torsion is absorbable* if there exists a different choice of π^a_i's such that the functions T^a_{ij} become zero, and otherwise that there is *torsion*. In the example at hand, since θ, dx, dy already span $T^*\Sigma$, the apparent torsion must be torsion. In notation that will be made invariant in §5.5, we let T denote the apparent torsion, i.e., the collection of functions T^a_{ij}, and we write $[T] = 0$ if the apparent torsion is absorbable.

Exercises 5.2.1:
1. Show that if $\tilde{\pi}^\epsilon = M^\epsilon_\delta \pi^\delta$ for some invertible matrix (M^ϵ_δ), then the new apparent torsion $\tilde{T}^a_{ij}$ is identically zero iff the original apparent torsion T^a_{ij} was identically zero. Conclude that the only possible way to absorb apparent torsion is to change the complement of the subspace $J \subset T^*\Sigma$ spanned by the π^ϵ's.

2. Show that if $\tilde{\pi}^\epsilon = \pi^\epsilon + M_a^\epsilon \theta^a$, then $\tilde{T}_{ij}^a = T_{ij}^a$. Conclude that the only possible way to absorb apparent torsion is to change the complement of $J/I \subset T^*\Sigma/I$ spanned by the π^ϵ's modulo I.
3. Show that at least in some cases one may absorb apparent torsion by changing the complement of the flag $J/I \subset T^*\Sigma/I$ by beginning with a system without torsion (e.g., the tautological system on $J^2(\mathbb{R}^2, \mathbb{R}^2)$) and making a "bad" initial choice of π^ϵ's such that there is apparent torsion.

The first step in our informal algorithm is to determine equations for the torsion, and the next step is to restrict to a submanifold $\Sigma' \subset \Sigma$ where the torsion is zero and check if the independence condition still holds.

We record these observations in the following flowchart:



Remark 5.2.2. The set defined by setting the torsion to zero may have many components and strata (as an analytic space). To find all integral manifolds to a system, one would have to restrict to each smooth stratum in turn.

5.3. Second example: constant coefficient homogeneous systems

Let $i : \Sigma \hookrightarrow J^1(V, W) \simeq V + W + V^* \otimes W$ be defined by a constant coefficient homogeneous system, as in Chapter 4. Let $J^1(V, W)$ have coordinates (x^i, u^a, p_i^a) and let $\theta^a = du^a - p_i^a dx^i$, $\omega^i = dx^i$. As usual, we omit the pullback i^* in the notation. The pullback of the contact system to Σ is still denoted

$$I = \{\theta^a\},$$
$$J = \{\theta^a, \omega^i\},$$

and we have

$$d\theta^a \equiv -dp_i^a \wedge \omega^i \mod I. \tag{5.2}$$

However, pulled back to Σ, the 1-forms dp_i^a are not all independent. Say we choose forms π^ϵ, where $\dim J + 1 \leq \epsilon \leq \dim \Sigma$, such that $\{\theta^a, \omega^i, \pi^\epsilon\}$ gives a local coframing of Σ. Since Σ is defined by homogeneous equations $B_a^{ri} p_i^a = 0$, we may choose the π^ϵ so that

$$dp_i^a = -A_{\epsilon i}^a \pi^\epsilon$$

for some constants $A_{\epsilon i}^a$ such that $B_a^{ri} A_{\epsilon i}^a = 0$. Thus we may rewrite (5.2) as

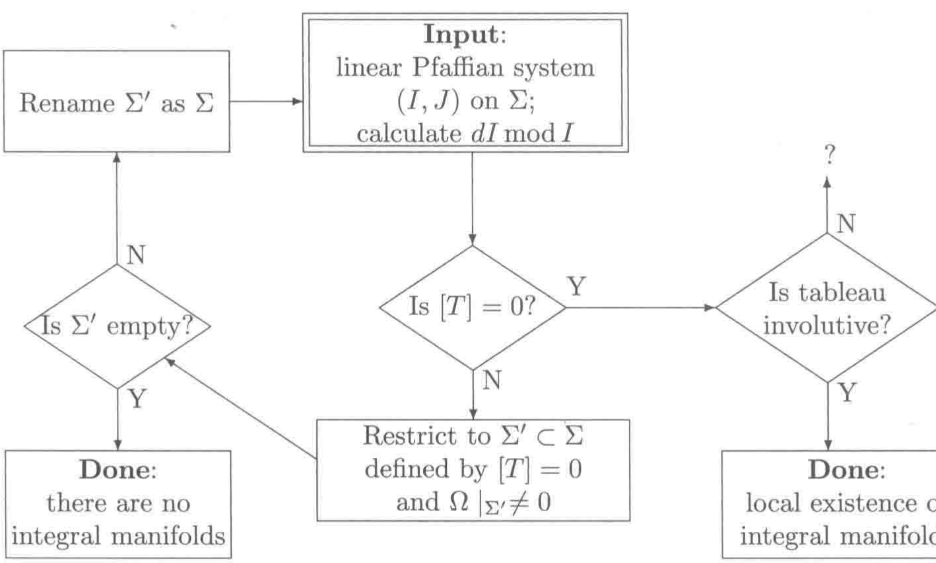
$$d\theta^a \equiv A_{\epsilon i}^a \pi^\epsilon \wedge \omega^i \mod I. \tag{5.3}$$

Observe that $(J/I)_x \simeq V^*$ and $I_x \simeq W^*$. When we work with general linear Pfaffian systems, these will be the definitions for V and W. The tableau is recovered by taking

$$A := \{A_{\epsilon i}^a v^i \otimes w_a \subseteq V^* \otimes W \mid \dim J + 1 \leq \epsilon \leq \dim \Sigma\}.$$

We saw in Chapter 4 that if the tableau A is involutive, we have local existence of integral manifolds, roughly depending on s_l functions of l variables, where l is the Cartan integer and s_l is the character of the tableau. For an arbitrary linear Pfaffian system with no torsion, we will define a tableau at a general point $x \in \Sigma$, and if this tableau is involutive, we will again have local existence of integral manifolds.

Our informal algorithm now looks like:



Remark 5.3.1. We can consider the more general case of a linear constant-coefficient but not necessarily homogeneous system. Say there are constants

C^r_i such that the equations are of the form

$$B^{ri}_a p^a_i = C^r_i x^i.$$

Then we must have

$$dp^a_i = -A^a_{\epsilon i}\pi^\epsilon + T^a_{ij}\omega^i,$$

where the $A^a_{\epsilon i}$ are as before, and T^a_{ij} are constants which satisfy $C^r_i = B^{rj}_a T^a_{ij}$. Now we have apparent torsion:

$$d\theta^a \equiv A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \bmod I.$$

As seen in 5.2.1, it might be possible to modify the π^ϵ, by adding multiples of the ω^j's, so that the T^a_{ij} all zero, i.e., so that the apparent torsion is absorbed. We examine the absorption of apparent torsion systematically in §5.5.

What if A is not involutive? If A is not involutive, we need to examine third-order information. The test will be to compare a naïve estimate for the dimension of $A^{(2)}$ with its actual dimension. (Recall from Chapter 4 that $A^{(2)}$ admits the interpretation as the space of admissible third-order terms in the Taylor series of a solution.)

We start with a new system, which is the pullback of the tautological linear Pfaffian system on

$$J^2(V,W) \simeq V + W + V^* \otimes W + W \otimes S^2 V^*$$

to $\tilde{\Sigma} := V + W + A + A^{(1)}$. This new system is called the *prolongation* of (I, J) on Σ.

Recall that, in coordinates $(x^i, u^a, p^a_i, p^a_{ij})$, the contact system on $J^2(V,W)$ is generated by

$$\theta^a = du^a - p^a_i dx^i,$$
$$\theta^a_i = dp^a_i - p^a_{ij} dx^j.$$

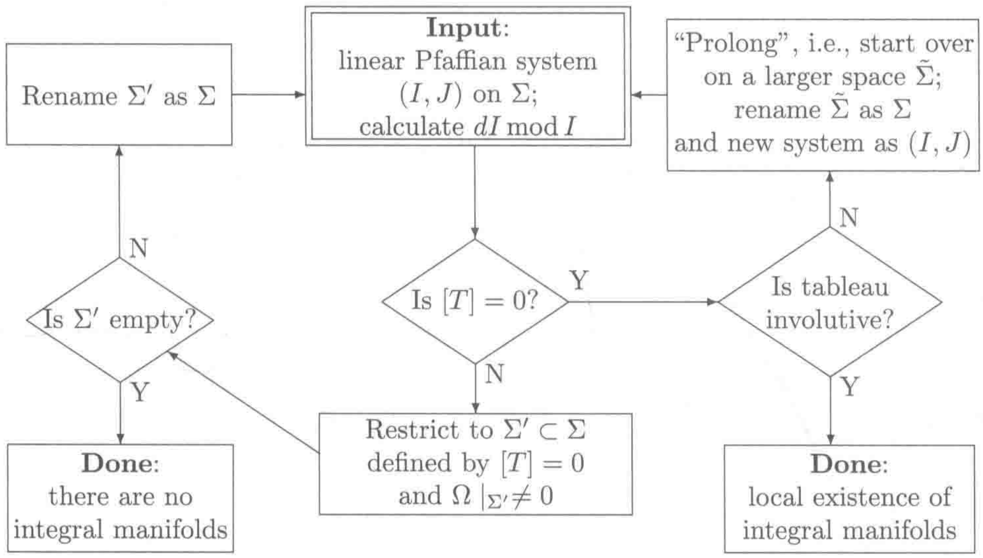
We abuse notation by writing θ^a, θ^a_i for $i^*(\theta^a), i^*(\theta^a_i)$, where

$$i : \tilde{\Sigma} \hookrightarrow V + W + V^* \otimes W + S^2 V^* \otimes W$$

is the inclusion. In particular, note that on $\tilde{\Sigma}$, only $\dim A$ of the forms θ^a_i are independent.

Writing our new system as $I = \{\theta^a, \theta^a_i\}$, we observe that $d\theta^a \equiv 0 \bmod I$, so our $(s + \dim A) \times n$ tableau, when represented by a matrix, will always have the first $s \times n$ block zero. We now apply Cartan's Test for involutivity to this tableau. If it is involutive, then local integral manifolds exist; if not, then further prolongation is necessary.

Here is our full informal algorithm:



Before formalizing this process, we illustrate it with a more substantial example:

5.4. The local isometric embedding problem

Let (M, g) and (N, h) be Riemannian manifolds. An embedding $f : M \to N$ is called *isometric* if $f^*(h) = g$.

Problem 5.4.1 (Analytic isometric embedding)**.** Let (M^n, g) be an analytic Riemannian manifold and let r be a positive integer. What (if any) are the local isometric embeddings $M^n \to \mathbb{E}^{n+r}$?

We will say that a manifold admits a *local isometric embedding* into Euclidean space if an isometric embedding exists on some neighborhood of any given point.

The above problem is stated vaguely—we will study it in the following form:

> Determine a function $r(n)$ with values in $\mathbb{Z}_+ \cup \infty$ such that every analytic Riemannian manifold of dimension n admits a local isometric embedding into $\mathbb{E}^{n+r}$.

While the answer to this question is known—in fact we will find $r(n)$ in §5.10—the corresponding question in the C^∞ category is still open.

In coordinates, choosing a metric corresponds to choosing $\binom{n+1}{2}$ functions of n variables subject to some open (nondegeneracy) conditions, and choosing a map $\mathbb{R}^n \to \mathbb{R}^{n+r}$ is the choice of $n + r$ functions of n variables.

So the determined case (the case where there are as many equations as unknowns) is when $r = \binom{n}{2}$. We will specialize to this case when we discuss the Cartan-Janet Theorem in §5.10.

As usual, instead of working on $M \times \mathbb{E}^{n+r}$, we will work on a larger space that takes into account the group actions. Let $\mathcal{F}_M$ and $\mathcal{F}_{\mathbb{E}^{n+r}}$ be the respective orthonormal frame bundles. Fix index ranges $1 \le i, j \le n$, $n+1 \le \mu, \nu \le n+r$. We use η to denote forms on $\mathcal{F}_M$ and ω to denote forms on $\mathcal{F}_{\mathbb{E}^{n+r}}$. Recall from §2.6 the structure equations on $\mathcal{F}_M$:

$$\begin{aligned} d\eta^j &= -\eta^j_k \wedge \eta^k, \\ d\eta^i_j &= -\eta^i_l \wedge \eta^l_j + \tfrac{1}{2} R^i_{jkl} \eta^k \wedge \eta^l, \end{aligned}$$

where the functions $R^i_{jkl} : \mathcal{F}_M \to \mathbb{R}$ are the coefficients of the Riemann curvature tensor. We set up an EDS whose integral manifolds are graphs $\Gamma \subset \mathcal{F}_{\mathbb{E}^{n+r}} \times \mathcal{F}_M$ of isometric embeddings $M^n \to \mathbb{E}^{n+r}$. As usual, we commit the standard abuse of notation, omitting pullbacks.

To simplify our calculations, we restrict to graphs Γ on which $\omega^\mu = 0$ and $\omega^i = \eta^i$, so $T_x M$ is spanned by $e_1, \ldots, e_n$, and the first n basis vectors of $T_x M$ and $T_x \mathbb{E}^{n+r}$ are aligned. On $\mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}}$, define the Pfaffian system

$$\mathcal{I} = \{\omega^\mu, \omega^j - \eta^j\}_{\mathsf{diff}}$$

with independence condition $\Omega = \eta^1 \wedge \ldots \wedge \eta^n$, so that

$$\begin{aligned} I &= \{\omega^\mu, \omega^j - \eta^j\}, \\ J &= \{\omega^\mu, \omega^j, \eta^j\}. \end{aligned}$$

We calculate

$$\begin{aligned} d\omega^\mu &= -\omega^\mu_j \wedge \omega^j - \omega^\mu_\nu \wedge \omega^\nu, \\ d(\omega^i - \eta^i) &= -\omega^i_j \wedge \omega^j - \eta^i_j \wedge \eta^j - \omega^i_\nu \wedge \omega^\nu. \end{aligned}$$

Thus,

$$\left.\begin{aligned} d\omega^\mu &\equiv 0 \\ d(\omega^i - \eta^i) &\equiv 0 \end{aligned}\right\} \bmod J,$$

and our system is a linear Pfaffian system, so we can begin the algorithm. We calculate

$$\left.\begin{aligned} d\omega^\mu &\equiv -\omega^\mu_j \wedge \eta^j \\ d(\omega^i - \eta^i) &\equiv -(\omega^i_j - \eta^i_j) \wedge \eta^j \end{aligned}\right\} \bmod I. \tag{5.4}$$

There is no torsion, since the forms $\omega^\mu_j, (\omega^i_j - \eta^i_j)$ are linearly independent. At a point $x \in \Sigma$, let $V^* = (J/I)_x$, $W^* = I_x$. We can use the structure equations (5.4), just as we used (5.3), to define the tableau

$$A = \begin{pmatrix} a^\mu_j \\ a^i_j \end{pmatrix},$$

where the entries are subject only to the relations $a^i_j + a^j_i = 0$. Here the π^ϵ's are the ω^μ_j and the $(\omega^i_j - \eta^i_j)$ for $i < j$. The Cartan characters (see Chapter 4) are given by $s_j = r + (n - j)$.

Write an element of $A^{(1)}$ as $(a^i_{jk} w_i + a^\mu_{jk} w_\mu) \otimes v^j v^k$. We see that $a^i_{jk} = -a^j_{ik} = a^i_{kj}$, so $a^i_{jk} = 0$ for all i, j, k. The only relations on the a^μ_{jk} are $a^\mu_{jk} = a^\mu_{kj}$. Thus $\dim A^{(1)} = r\binom{n+1}{2} < s_1 + 2s_2 + \ldots + ns_n$, and the tableau is not involutive.

We continue our algorithm, starting over on a larger space where the second derivatives are included as independent variables. In this case, we will call these new variables h^μ_{ij}. (They will turn out to be the coefficients of the second fundamental form.) We start over on the manifold

$$\tilde{\Sigma} = \mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}} \times \mathbb{R}^{r\binom{n+1}{2}}.$$

The system $(\tilde{I}, \tilde{J})$ defined by

$$(5.5) \qquad \begin{aligned} \tilde{I} &= \{\omega^\mu, \omega^j - \eta^j, \omega^i_j - \eta^i_j, \omega^\mu_j - h^\mu_{jk}\eta^k\}, \\ \tilde{J} &= \{\omega^\mu, \omega^j, \eta^j, \omega^i_j - \eta^i_j, \omega^\mu_j\} \end{aligned}$$

is the *prolongation* of (I, J).

As with the prolongation in the previous example, we have

$$\left.\begin{aligned} d\omega^\mu &\equiv 0 \\ d(\omega^i - \eta^i) &\equiv 0 \end{aligned}\right\} \bmod \tilde{I}.$$

We continue our computations:

$$d(\omega^i_j - \eta^i_j) \equiv -\tfrac{1}{2}(R^i_{jkl} - \sum_\mu (h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk}))\eta^k \wedge \eta^l \bmod \tilde{I}.$$

Here we are presented with the same situation as in §5.2: there is torsion, and so there are no integral elements except at points of $\tilde{\Sigma}$ where

$$(5.6) \qquad R^i_{jkl} - \sum_\mu (h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk}) = 0.$$

Thus we start over again on the submanifold $\Sigma' \subset \tilde{\Sigma}$ defined by (5.6). Since (5.6) is nothing but the Gauss equations (see Chapter 2), it is clear in hindsight that we should have begun working on the submanifold where (5.6) holds. Assume for the moment that the pullback to Σ' of our independence condition is nonvanishing, so that we may go on to calculate our new tableau:

$$(5.7) \qquad d(\omega^\mu_j - h^\mu_{jk}\omega^k) \equiv (-dh^\mu_{jk} - h^\nu_{jk}\omega^\mu_\nu + h^\mu_{jl}\omega^l_k + h^\mu_{kl}\omega^l_j) \wedge \omega^k \bmod \tilde{I}.$$

To simplify notation, let $\pi^\mu_{jk} = -dh^\mu_{jk} - h^\nu_{jk}\omega^\mu_\nu + h^\mu_{jl}\omega^l_k + h^\mu_{kl}\omega^l_j$. On $\tilde{\Sigma}$, the forms π^μ_{jk} are all independent, but when we restrict to Σ', they satisfy relations obtained by differentiating (5.6).

The next step is to check the torsion. First, we restrict to a simple, special case. Other cases will be treated below and in §5.10.

Case $n = 2, r = 1$. The only independent curvature component is $R_{1212} = R$, and equation (5.6) becomes

$$R = h_{11}h_{22} - (h_{12})^2, \tag{5.8}$$

where we have suppressed the $\mu = 3$ index on h.

Let $\theta_j = \omega^3_j - h_{jk}\omega^k$. Then (5.7) specializes to

$$d\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \equiv \begin{pmatrix} -dh_{11} + 2h_{12}\eta^2_1 & -dh_{12} - (h_{11} - h_{22})\eta^2_1 \\ -dh_{12} - (h_{11} - h_{22})\eta^2_1 & -dh_{22} - 2h_{12}\eta^2_1 \end{pmatrix} \wedge \begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} \mod \tilde{I}.$$

Since $R : \mathcal{F}_M \to \mathbb{R}$ is constant on the fibers of the projection to M, we may write $dR = R_1\eta^1 + R_2\eta^2$. Differentiating (5.8) gives

$$R_1\eta^1 + R_2\eta^2 - h_{11}dh_{22} - h_{22}dh_{11} + 2h_{12}dh_{12} = 0. \tag{5.9}$$

Upon setting

$$\begin{aligned}
\pi_1 &= -dh_{11} + 2h_{12}\eta^2_1,\\
\pi_2 &= -dh_{12} - (h_{11} - h_{22})\eta^2_1,\\
\pi_3 &= -dh_{22} - 2h_{12}\eta^2_1,
\end{aligned}$$

the relation (5.9) becomes

$$R_1\eta^1 + R_2\eta^2 + h_{11}\pi_3 + h_{22}\pi_1 - 2h_{12}\pi_2 = 0. \tag{5.10}$$

Aside 5.4.2. The seeming miracle that the η^2_1 terms all cancel is due to the fact that the form η^2_1 is dual to a *Cauchy characteristic vector field* (see Chapter 6). Intuitively, integral curves of this vector field correspond to spinning the frame within the tangent space of the surface, which doesn't change the embedding.

In fact, Theorem 6.1.20 implies that our EDS actually lives on the quotient of Σ by this foliation. Since integral manifolds of the EDS exist if and only if torsion vanishes on the quotient manifold, we expect that the torsion upstairs will not involve η^2_1.

In n dimensions, the forms η^i_j are dual to Cauchy characteristics for the isometric embedding system. Rather than ignoring them, we could have incorporated them into the independence condition. Then integral manifolds would correspond to graphs of embeddings of $\mathcal{F}_M$ in $\mathcal{F}_{\mathbb{E}^{n+r}}$ which are lifts of isometric embeddings $M \to \mathbb{E}^{n+r}$.

Of the forms π_1, π_2, π_3, two are independent modulo J/I. We solve for π_3 using (5.10) and plug in:

$$d\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \equiv \begin{pmatrix}\pi_1 & \pi_2\\ \pi_2 & \frac{1}{h_{11}}(h_{22}\pi_1 - 2h_{12}\pi_2)\end{pmatrix} \wedge \begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} - \begin{pmatrix}0\\ \frac{R_1}{h_{11}}\eta^1\wedge\eta^2\end{pmatrix}.$$

Here we have isolated the apparent torsion term.

As mentioned in 5.3.1, it might be possible to eliminate the apparent torsion by choosing a different complement to $J \subset T^*\Sigma$. Here, if we let

$$\tilde{\pi}_1 = \pi_1 - \frac{R_1}{h_{22}}\eta^1,$$

then $\tilde{\pi}_1$ is still independent of η^1, η^2, and the equations become

$$d\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \equiv \begin{pmatrix}\tilde{\pi}_1 & \pi_2\\ \pi_2 & \frac{1}{h_{11}}(h_{22}\tilde{\pi}_1 - 2h_{12}\pi_2)\end{pmatrix} \wedge \begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}. \tag{5.11}$$

We have absorbed the apparent torsion by changing our choice of coframing.

We can now perform Cartan's Test on the tableau $A^{(1)}$ associated to (5.7). Here $s_1 = 2, s_2 = 0$, and $s_1 + 2s_2 = 2 = \dim A^{(2)}$, so the tableau is involutive according to Definition 4.5.4. Invoking the Cartan-Kähler Theorem 5.5.6 below, we obtain

Theorem 5.4.3. *Let (M^2, g) be an analytic Riemannian manifold of dimension two. Then there exist local isometric embeddings $M^2 \to \mathbb{E}^3$, and such embeddings depend on two functions of one variable.*

Note that there are actually two cases for the theorem, one assuming that it is possible to have $h_{11} \neq 0$, and the second being the flat case where at least we know existence. Dependence on two functions of one variable only applies to the first case.

Exercise 5.4.4: Verify directly that $\dim A^{(2)} = 2$.

Aside 5.4.5 (Another easy case)**.** Suppose that the codimension r is small enough that $\frac{n^2(n^2-1)}{12} > \binom{n+1}{2}r$. To see if (5.6) is solvable, let $V = \mathbb{R}^n$, $W = (\mathbb{R}^r, \langle\,,\rangle)$, and define

$$\gamma : S^2V^* \otimes W \to \mathcal{K}(V),$$
$$h^a_{ij}v^iv^j \otimes w_a \mapsto \sum_a (h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk})v^i \otimes v^j \otimes v^k \otimes v^l.$$

Here, $\mathcal{K}(V) = S_{22}(V)$ is the kernel of the skew-symmetrization map $\delta : \Lambda^2V \otimes \Lambda^2V \to \Lambda^4V$ (see Appendix A), and $\dim \mathcal{K}(V) = \frac{n^2(n^2-1)}{12}$. Our assumption implies that γ cannot be surjective. So, only curvature tensors taking value in a subvariety of $\mathcal{K}(V)$ can satisfy (5.6). Thus, there are no isometric embeddings of a 'generic' metric in this range of codimensions.

5.5. The Cartan algorithm formalized: tableau, torsion and prolongation

Let $I = \{\theta^a\}$, $J = \{\theta^a, \omega^i\}$ be a linear Pfaffian system on a manifold Σ. Choose a complement $\{\pi^\epsilon\}$ to J, where $1 \leq \epsilon \leq \dim\Sigma - \dim J$, to obtain a

local coframing of Σ. By Exercise 5.1.2 there exist functions $A^a_{\epsilon i}$, T^a_{ij} such that the derivatives of forms in I may be written

$$d\theta^a \equiv A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \bmod I.$$

The functions $A^a_{\epsilon i}, T^a_{ij}$ depend on our choices of bases for I and J. We now construct invariants from them.

Tableaux. Fix a generic point $x \in \Sigma$. Let $V^* = (J/I)_x$ and $W^* = I_x$. Write $w^a = \theta^a_x$, $v^j = \omega^j_x$, and define the *tableau of* (I, J) *at* x by

$$A_x := \{A^a_{i\epsilon} w_a \otimes v^i | 1 \leq \epsilon \leq \dim \Sigma - \dim J_x\} \subseteq W \otimes V^*.$$

Exercise 5.5.1: Verify that A_x is independent of any choices.

Remark 5.5.2. The tableau can also be defined by *symbol relations.* For, if $d\theta^a \equiv \pi^a_i \wedge \omega^i \bmod I$, write

$$B^{ri}_a \pi^a_i \equiv C^r_j \omega^j \bmod I \quad \forall 1 \leq r \leq \operatorname{codim} A.$$

Then $B = \{B^{ri}_a v_i \otimes w^a\} \subset V \otimes W^*$ is the annihilator of A.

Example 5.5.3 (Second-order PDE)**.** Recall how a second-order PDE for one function of two variables is expressed as an EDS. We use the classical coordinates (x, y, z, p, q, r, s, t) on $J^2(\mathbb{R}^2, \mathbb{R})$ and contact forms

$$\begin{aligned} \theta_0 &= dz - p\,dx - q\,dy, \\ \theta_1 &= dp - r\,dx - s\,dy, \\ \theta_2 &= dq - s\,dx - t\,dy. \end{aligned}$$

(In particular, p, q, r, s, t correspond respectively to the jet coordinates p_1, p_2, p_{11}, p_{12}, p_{22} defined in §1.9.) Suppose the PDE takes the form

$$F(x, y, z, p, q, r, s, t) = 0. \tag{5.12}$$

Assume that $\{F = 0\}$ defines a smooth submanifold $\Sigma^7 \subset J^2(\mathbb{R}^2, \mathbb{R})$, and that at least one of F_r, F_s, F_t is nonzero at each point of Σ, and say that F is *regular* when these assumptions hold. Regularity ensures that $\theta_0, \theta_1, \theta_2$ restrict to be linearly independent on Σ, and that the projection $\pi : \Sigma \to J^1(\mathbb{R}^2, \mathbb{R})$ is a smooth submersion at each point.

Let I be the Pfaffian system spanned by the restrictions of $\theta_0, \theta_1, \theta_2$ to Σ. Since $d\theta_0 \equiv 0$ modulo I, the ideal $\mathcal{I} = \{I\}_{\mathsf{diff}}$ is generated algebraically by I and the 2-forms

$$d\theta_1 = -(dr \wedge dx + ds \wedge dy), \qquad d\theta_2 = -(ds \wedge dx + dt \wedge dy).$$

Near any point in Σ, one can obtain 1-forms π_1, π_2, π_3 such that

$$\left.\begin{aligned} d\theta_1 &\equiv -(\pi_1 \wedge dx + \pi_2 \wedge dy) \\ d\theta_2 &\equiv -(\pi_2 \wedge dx + \pi_3 \wedge dy) \end{aligned}\right\} \bmod I \tag{5.13}$$

and such that π_1, π_2, π_3 satisfy the symbol relation

$$F_r\pi_1 + F_s\pi_2 + F_t\pi_3 \equiv 0 \mod I. \tag{5.14}$$

For example, if $F_r \neq 0$, we may choose

$$\begin{aligned}
\pi_1 &= dr + \left(\frac{F_r D_x F - F_s D_y F}{F_r^2}\right) + \frac{D_y F}{F_r} dy,\\
\pi_2 &= ds + \frac{D_y F}{F_r} dx,\\
\pi_3 &= dt,
\end{aligned}$$

where $D_x F = F_x + pF_z + rF_p + sF_q$ and $D_y F = F_y + qF_z + sF_p + tF_q$.

Let $u \in \Sigma$ and let $E \in \mathcal{V}_2(\mathcal{I}, \Omega)$ be an integral 2-plane at u. If we use the bases $\{\theta_0, \theta_1, \theta_2\}$ for $W^* = I_u$ and $\{dx, dy\}$ for $E^* \cong V^* = (J/I)_u$, then (5.14) shows that the tableau takes the form

$$A_u = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \\ b & c \end{pmatrix} \middle| \, aF_r(u) + bF_s(u) + cF_t(u) = 0 \right\} \subset W \otimes V^*. \tag{5.15}$$

Torsion. Recall that the terms $T^a_{ij}\omega^i \wedge \omega^j$ in (5.1) are called *apparent torsion*. If it is possible to choose new $\tilde{\pi}^\epsilon$'s such that $\tilde{T}^a_{ij} = 0$, we say the apparent torsion is *absorbable*. If it is not possible, we will say there is *torsion*. In other words, a particular choice of complement to $J_x \subset T^*_x\Sigma$ may yield apparent torsion at x, while we say there is torsion at x if all choices yield apparent torsion. In that case there are no integral elements at x. We now discuss these statements without reference to bases.

The change in the T^a_{ij} that a change of complement $\tilde{\pi}^\epsilon = \pi^\epsilon + e^\epsilon_i\omega^i$ effects is

$$\widetilde{T}^a_{ij} = T^a_{ij} + (A^a_{j\epsilon}e^\epsilon_i - A^a_{i\epsilon}e^\epsilon_j).$$

Let $E^a_{ij} = A^a_{j\epsilon}e^\epsilon_i - A^a_{i\epsilon}e^\epsilon_j$ and let

$$\delta : W \otimes V^* \otimes V^* \to W \otimes \Lambda^2 V^*$$

be the skew-symmetrization map. The space of possible modifications to apparent torsion is $\delta(A \otimes V^*)$, because $E := E^a_{ij} w_a \otimes v^i \otimes v^j$ lies in $\delta(A \otimes V^*)$. Then the *torsion* of (I, J) at x is defined to be

$$[T] := [T^a_{ij} w_a \otimes v^i \wedge v^j] \in W \otimes \Lambda^2 V^* / \delta(A_x \otimes V^*).$$

We will use the notation $H^{0,2}(A) = W \otimes \Lambda^2 V^* / \delta(A_x \otimes V^*)$. (This is one of the Spencer cohomology groups, which will be discussed in §5.7.)

Exercise 5.5.4: Compute the C^r_j of Remark 5.5.2 in terms of the coefficients $A^a_{\epsilon i}, T^a_{ij}$.

If $[T] \equiv 0$, then any apparent torsion terms we have for a specific choice of π^ϵ's are absorbable and we could choose new forms $\tilde{\pi}^\epsilon$'s to get all the apparent torsion to disappear. For the purposes of Cartan's Test, this choice is not necessary because the tableau is independent of the choice of π^ϵ by Exercise 5.5.1. If $[T(x)]$ is not identically zero, we need to restrict our system to $\Sigma' \subset \Sigma$ on which $[T] \equiv 0$. On this subset we may obtain new symbol relations resulting from $d[T] = 0$.

It is possible that the equations $d[T] = 0$ force a relation among the ω^i on an open subset of Σ'. If this happens, we must restrict further to the subset $\Sigma'' \subset \Sigma'$ where the ω^i are independent. If Σ'' is empty, there are no integral manifolds and we are done. In any event, restricting to Σ' and Σ'' will in general introduce new symbol relations, which in turn could introduce more apparent torsion, and so we must repeat the above process.

Exercises 5.5.5:

1. Say we have an isomorphism $V \to W$ such that $A \simeq \Lambda^2 V$. Show that $H^{0,2}(A) = 0$. Compare this to the fundamental lemma of Riemannian geometry in Chapter 2. What about if $A \simeq S^2 V$?
2. Let (I, J) be a linear Pfaffian system on Σ. Let $x \in \Sigma$ and suppose $\dim(J/I)_x = \dim I_x = 2$. Show that if $s_1(A_x) = 2$, then $[T]_x = 0$.

The Cartan-Kähler Theorem for Linear Pfaffian systems.

Theorem 5.5.6. *Let (I, J) be an analytic linear Pfaffian system on a manifold Σ, let $x \in \Sigma$ be a point and let U be a neighborhood containing x, such that for all $y \in U$,*

1. *$[T]_y = 0$, and*
2. *the tableau A_y is involutive.*

Then solving a series of Cauchy problems yields analytic integral manifolds to (I, J) passing through x that depend (in the sense discussed in Chapter 4) on s_l functions of l variables, where s_l is the character of the system.

We interpret the Cartan-Kähler Theorem as saying: if there is no obstruction to having an integral element at a general point $x \in \Sigma$, then the space of integral manifolds passing through x is the same size as the space of integral manifolds for the corresponding linearized problem. In other words, if the linear algebra at the infinitesimal level works our right, everything works out right.

When a linear Pfaffian system (I, J) satisfies the conditions of Theorem 5.5.6 at x, we say it is *involutive at x*. (Note that because $x \in \Sigma$ is a *general point*, we only need to check conditions 1 and 2 at x, and it will be guaranteed that they are satisfied at nearby points.)

We will give a complete proof of a more general version of the Cartan-Kähler Theorem in Chapter 7. The present version could be proved by choosing local coordinates and doing estimates comparing a formal solution to the system with its linearization.

Finally, we formalize the process of prolongation:

Prolongation. Let $\mathcal{I}$ be an arbitrary EDS defined on a manifold Σ. Consider $\pi : \mathbf{G}(n, T\Sigma) \to \Sigma$, the *Grassmann bundle* of all n-planes in all tangent spaces to Σ. We denote points of $\mathbf{G}(n, T\Sigma)$ by (p, E), where $p \in \Sigma$ and $E \subset T_p\Sigma$ is an n-dimensional subspace. Much like the contact system on the space of k-jets, $\mathbf{G}(n, T\Sigma)$ carries a canonical linear Pfaffian system on it, whose integral manifolds are exactly the lifts $\tilde{X}$ of immersed submanifolds $X^n \subset \Sigma$ to $\mathbf{G}(n, T\Sigma)$ defined by $\tilde{x} = (x, T_xX)$. The system on $\mathbf{G}(n, T\Sigma)$ is defined by

$$I_{(p,E)} := \pi^*(E^\perp), \qquad J_{(p,E)} := \pi^*(T_p^*\Sigma).$$

Now let $\mathcal{V}_n(\mathcal{I}) \subset \mathbf{G}(n, T\Sigma)$ be the subbundle (more precisely, subsheaf) of n-dimensional integral elements to $\mathcal{I}$, whose fiber over a point $x \in \Sigma$ is $\mathcal{V}_n(\mathcal{I})_x$ as defined in §1.9. That is,

$$\mathcal{V}_n(\mathcal{I}) := \{(x, E) \in \mathbf{G}(n, T\Sigma) \mid \mathcal{I}_x|_E = 0\}.$$

The *prolongation* of $\mathcal{I}$ is the restriction of the canonical system on $\mathbf{G}(n, T\Sigma)$ to $\mathcal{V}_n(\mathcal{I})$. (We should really call this the prolongation of $\mathcal{I}$ for n-dimensional integral manifolds.) If $\mathcal{I}$ comes equipped with an independence condition Ω, we recall that

$$\mathcal{V}_n(\mathcal{I}, \Omega) := \{(x, E) \in \mathbf{G}(n, T\Sigma) \mid \Omega_x(E) \neq 0 \text{ and } \mathcal{I}_x|_E = 0\}.$$

In this case we define the prolongation to be the restriction of the canonical system to $\mathcal{V}_n(\mathcal{I}, \Omega)$.

5.6. Summary of Cartan's algorithm for linear Pfaffian systems

The final version of our flowchart appears on page 178.

Remark 5.6.1. The Cartan algorithm will *not* necessarily yield all integral manifolds of the original system, only the integral manifolds arising from well-posed Cauchy problems at general points.

Remark 5.6.2. Each time one prolongs, there may be many different components of $\mathcal{V}_n(\mathcal{I})$ to restrict to. To find all possible integral manifolds, one must carry out the algorithm on each component. The Cartan-Kuranishi Prolongation Theorem [**20**] says in effect that this process terminates eventually, but gives no hint of how long it will take.

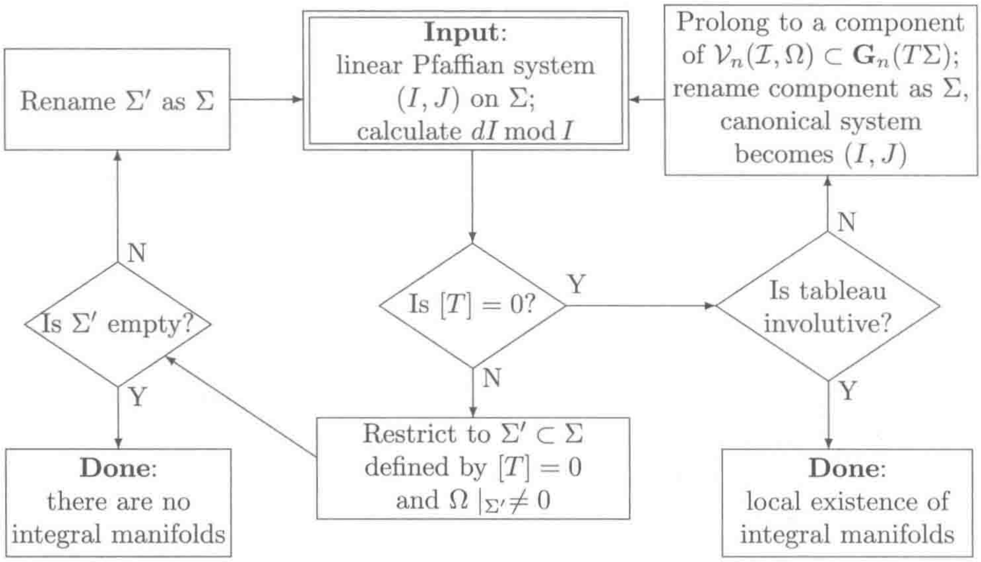


Figure 1. Our final flowchart.

Warning: Where the algorithm ends up (i.e., in which "Done" box in the flowchart) may depend on the component one is working on.

Summary. Let (I, J) be a linear Pfaffian system on Σ. We summarize Cartan's algorithm:

(1) Take a local coframing of Σ adapted to the filtration $I \subset J \subset T^*\Sigma$. Let $x \in \Sigma$ be a general point. Let $V^* = (J/I)_x$, $W^* = I_x$, and let $v^i = \omega^i_x, w^a = \theta^a_x$ and v_i, w_a be the corresponding dual basis vectors.

(2) Calculate $d\theta^a$; since the system is linear, these are of the form
$$d\theta^a \equiv A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j \bmod I.$$
Define the *tableau* at x by
$$A = A_x := \{A^a_{\epsilon i} v^i \otimes w_a \subseteq V^* \otimes W \mid 1 \leq \epsilon \leq r\} \subseteq W \otimes V^*.$$
Let δ denote the natural skew-symmetrization map $\delta : W \otimes V^* \otimes V^* \to W \otimes \Lambda^2 V^*$ and let
$$H^{0,2}(A) := W \otimes \Lambda^2 V^* / \delta(A \otimes V^*).$$
The *torsion* of (I, J) at x is
$$[T]_x := [T^a_{ij} w_a \otimes v^i \wedge v^j] \in H^{0,2}(A).$$

(3) If $[T]_x \neq 0$, then start again on $\Sigma' \subset \Sigma$ defined by the equations $[T] = 0$, with the additional requirement that J/I has rank n over

Σ'. Since the additional requirement is a transversality condition, it will be generically satisfied as long as $\dim \Sigma' \geq n$. In practice one works infinitesimally, using the equations $d[T] = 0$, and checks what relations $d[T] \equiv 0 \mod I$ imposes on the forms π^ϵ used before.

(4) Assume $[T]_x = 0$. Let $A_k := A \cap (\mathrm{span}\{v^{k+1}, \ldots, v^n\} \otimes W)$. Let $A^{(1)} := (A \otimes V^*) \cap (W \otimes S^2 V^*)$, the *prolongation* of the tableau A. Then

$$\dim A^{(1)} \leq \dim A + \dim A_1 + \ldots + \dim A_{n-1}$$

and A is *involutive* if equality holds.

Warning: One can fail to obtain equality even when the system is involutive if the bases were not chosen sufficiently generically. In practice, one does the calculation with a natural, but perhaps nongeneric basis and takes linear combinations of the columns of A to obtain genericity. If the bases are generic, then equality holds iff A is involutive.

When doing calculations, it is convenient to define the *characters* s_k by $s_1 + \ldots + s_k = \dim A - \dim A_k$, in which case the inequality becomes $\dim A^{(1)} \leq s_1 + 2s_2 + \ldots + ns_n$. If $s_p \neq 0$ and $s_{p+1} = 0$, then s_p is called the *character* of the tableau and p the *Cartan integer*. If A is involutive, then the Cartan-Kähler Theorem applies, and one has local integral manifolds depending on s_p functions of p variables.

(5) If A is not involutive, prolong, i.e., start over on the pullback of the canonical system on the Grassmann bundle to the space of integral elements. In calculations this amounts to adding the elements of $A^{(1)}$ as independent variables, and adding differential forms $\theta^a_i := A^a_{\epsilon i}\pi^\epsilon - p^a_{ij}\omega^j$ to the ideal, where $p^a_{ij}v^i v^j \otimes w_a \in A^{(1)}$.

5.7. Additional remarks on the theory

Another interpretation of $A^{(1)}$. We saw in Chapter 4 that for a constant-coefficient, first-order, homogeneous system defined by a tableau A, the prolongation $A^{(1)}$ is the space of admissible second-order terms $p^a_{ij}x^i x^j$ in a power series solution of the system. This was because the constants p^a_{ij} had to satisfy $p^a_{ij}w_a \otimes v^i \otimes v^j \in A^{(1)}$.

The following proposition, which is useful for computing $A^{(1)}$, is the generalization of this observation to linear Pfaffian systems:

Proposition 5.7.1. *After fixing $x \in \Sigma$ and a particular choice of 1-forms $\pi^a_i \mod I$ satisfying $d\theta^a \equiv \pi^a_i \wedge \omega^i \mod I$, $A^{(1)}$ may be identified with the space of 1-forms $\tilde{\pi}^a_i \mod I$ satisfying $d\theta^a \equiv \tilde{\pi}^a_i \wedge \omega^i \mod I$, as follows: any such*

set $(\tilde{\pi}_i^a)$ *may be written as* $\tilde{\pi}_i^a \equiv \pi_i^a + p_{ij}^a \omega^j \bmod I$, *with* $p_{ij}^a w_a \otimes v^i v^j \in A^{(1)}$. *In particular,* $A^{(1)}$ *is an affine space.*

For example, in Chapter 4, the initial choice was $\pi_i^a \equiv 0$ and the other choices were $p_{ij}^a dx^j$.

Proof. Let $\tilde{\pi}_i^a \equiv \pi_i^a + p_{ij}^a \omega^j \bmod I$. The $\tilde{\pi}_i^a$ must satisfy the symbol relations $B_a^{ri}\tilde{\pi}_i^a \equiv 0 \bmod I$. But $B_a^{ri}\tilde{\pi}_i^a \equiv B_a^{ri}(\pi_i^a + p_{ij}^a\omega^j) \equiv B_a^{ri}(p_{ij}^a\omega^j) \bmod I$, since π_i^a also satisfies the symbol relations. This implies $B_a^{ri}p_{ij}^a = 0$ for all j, since the ω^j are independent, so $p_{ij}^a w_a \otimes v^i \otimes v^j \in A \otimes V^*$. But we also need $\tilde{\pi}_i^a \wedge \omega^i \equiv \pi_i^a \wedge \omega^j + p_{ij}^a \omega^i \wedge \omega^j \equiv 0 \bmod I$. We already have $\pi_i^a \wedge \omega^i \equiv 0 \bmod I$, so this forces $p_{ij}^a \omega^i \wedge \omega^j = 0$, which implies $p_{ij}^a = p_{ji}^a$. □

Remark 5.7.2. If there is no torsion, $A_x^{(1)}$ is the space of integral elements at x. After fixing an integral element, it becomes a linear space. This is the reason for the adjective "linear" in the name linear Pfaffian systems.

Invariants of tableaux. Here we define a collection of invariants of a tableau. We will only use the invariants $H^{j,2}$, but we might as well define them all.

Definition 5.7.3. The *Spencer cohomology groups* $H^{i,j}(A)$ of a tableau $A \subset W \otimes V^*$ are defined as follows: Let

$$\delta_j : (W \otimes S^d V^*) \otimes \Lambda^j V^* \to W \otimes S^{d-1} V^* \otimes \Lambda^{j+1} V^*$$

be defined by

$$\delta_j(f \otimes \xi) = df \wedge \xi,$$

where, for $f \in W \otimes S^d V^*$, $\xi \in \Lambda^j V^*$, we define df by considering f as a W-valued function on V, and extend δ_j by linearity. Note that $\delta_j(A^{(i)} \otimes \Lambda^j V^*) \subseteq A^{(i-1)} \otimes \Lambda^{j+1} V^*$.

Define

$$H^{i,j}(A) := \frac{\ker \delta_j(A^{(i-1)} \otimes \Lambda^j V^*)}{\operatorname{Image} \delta_{j-1}(A^{(i)} \otimes \Lambda^{j-1} V^*)}.$$

Let $\tilde{W} = A \oplus W$. Then $A^{(1)}$ can be considered as a tableau in $\tilde{W} \otimes V^*$ with the $W \otimes V^*$ block zero. Consider:

$$\begin{aligned} H^{0,2}(A^{(1)}) &= \frac{\tilde{W} \otimes \Lambda^2 V^*}{\delta(A^{(1)} \otimes V^*)} \\ &= \frac{W \otimes V^* \otimes \Lambda^2 V^*}{\delta(((A \otimes V^*) \cap (W \otimes S^2 V^*)) \otimes V^*)} \\ &= H^{1,2}(A). \end{aligned}$$

Exercise 5.7.4: Show that
$$H^{0,2}(A^{(p)}) = H^{p,2}(A).$$

The *Kuranishi Prolongation Theorem* (see [**20**]) implies that after a finite number of prolongations a tableau will become involutive (including the possibility that it becomes empty). This implies

Theorem 5.7.5 (Goldschmidt version of Cartan-Kähler [**61**]). *Let (I, J) be a linear Pfaffian system on an analytic manifold Σ and let $x \in \Sigma$ be a general point. If $H^{p,2}(A_x) = 0$ for all p, then there exist integral manifolds passing through x.*

This theorem is useful because sometimes one can show that the groups $H^{p,2}(A_x)$ are zero without actually calculating the prolongations (see [**57**]).

More generally, it is sufficient to show that the corresponding vector bundles with induced torsion sections are such that the sections are zero; see Chapter 8 for the construction of the associated vector bundles.

Characteristic Hyperplanes. Let (I, J) be a linear Pfaffian system on a manifold Σ with no torsion, and let $E \in \mathcal{V}_n(I, J)_x$. Let $H \subset E$ be a hyperplane. We address the following question:

Under what circumstances is E the only integral n-plane that contains H?

Definition 5.7.6. A hyperplane $H \subset E \subset T_x\Sigma$ is said to be *characteristic* if it has more than one extension to an n-dimensional integral element.

With notation as above, we may choose the coframe so that
$$\theta^a|_E = \pi^a_i|_E = 0.$$
Assume moreover that E is uniquely defined by these equations. (This amounts to assuming that $\{\theta^a, \omega^i, \pi^a_i\}$ span the cotangent space to Σ, i.e., that there are no Cauchy characteristics.)

We can also choose frames such that $\omega^n|_H = 0$. Let $e_1, \dots, e_n$ be a basis for E dual to the ω^i. Then $v \in T_xM$, $v \notin H$, completes H to an integral n-plane if and only if

$$(v, e_1, \dots, e_{n-1}) \lrcorner \Phi = 0 \tag{5.16}$$

for all $\Phi \in \mathcal{I}^n$. In fact, it is sufficient to require that $v \lrcorner \theta^i = 0$ and to verify (5.16) for $\Phi = \pi^a_i \wedge \omega^J$, where J is any multi-index of length $n-1$ containing the index i. Since $e_i \lrcorner \pi^\epsilon = 0$, (5.16) becomes

$$v \lrcorner \pi^a_\alpha = 0, \qquad \forall a, \alpha, \quad 1 \le \alpha < n. \tag{5.17}$$

Now recall the symbol mapping from Definition 4.6.2, defining the characteristic variety Ξ_A for $A = A_x$. If $\xi_H \in \mathbb{P}E^*$ corresponds to $H \subset E$, then

$\xi_H \in \Xi_A$ if and only if

$$B_a^{rn} w^a = 0$$

for some nonzero vector $w \in W$. In other words, $\xi_H \in \Xi_A$ if and only if the tableau contains a matrix with the first $n-1$ columns zero and the last column nonzero, or equivalently, some linear combination of the π_n^a is linearly independent from the forms in the first $n-1$ columns, so that v is not uniquely determined by (5.17). We have now proved

Theorem 5.7.7. *A hyperplane $H \subset E \subset T_x\Sigma$ is characteristic iff the corresponding element $\xi_H \in \mathbb{P}E^*$ is such that $\xi_H \in \Xi_A$.*

Example (5.5.3 continued)**.** Let $\Sigma \subset J^2(\mathbb{R}^2, \mathbb{R})$ be the submanifold defined by a regular second-order PDE and let A be the tableau (5.15) at a point $u \in \Sigma$. Then a nonzero covector $\xi = \xi_1 dx + \xi_2 dy \in V^*$ belongs to Ξ_A if and only if the map $\sigma_\xi(w) := w \otimes \xi \mod A$ has a nonzero kernel. Write

$$w \otimes \xi = \begin{pmatrix} w_1\xi_1 & w_1\xi_2 \\ w_2\xi_1 & w_2\xi_2 \\ w_3\xi_1 & w_3\xi_2 \end{pmatrix}.$$

Then $w \otimes \xi \in A$ if and only if $w_1 = 0$, $w_2\xi_2 = w_3\xi_1$ and $F_r w_2\xi_1 + F_s w_2\xi_2 + F_t w_3\xi_2 = 0$. This homogeneous system of linear equations for the w_a has a nontrivial solution if and only if

$$F_r\xi_1^2 + F_s\xi_1\xi_2 + F_t\xi_2^2 = 0. \tag{5.18}$$

A linear subspace of an integral plane E is said to be *characteristic* if it is annihilated by some covector $\xi \in \Xi_A$. Thus, the characteristic hyperplanes in E are null lines for the quadratic form defined by the symmetric matrix

$$\begin{pmatrix} F_r & \frac{1}{2}F_s \\ \frac{1}{2}F_s & F_t \end{pmatrix}$$

with respect to the basis dual to dx, dy.

5.8. Examples

The examples with Roman numerals are adapted from Cartan's 1945 treatise [**31**].

Example 5.8.1 (The tautological system for surfaces in $\mathbb{E}^3$, defined on $ASO(3) \times \mathbb{R}^3{}_{h_{ij}}$)**.** The integral manifolds of this system are the first-order adapted lifts of surfaces in $\mathbb{R}^3$ (see §2.1), together with the components of the second fundamental form.

Let $I = \{\omega^3, \theta^3_j := \omega^3_j - h_{jk}\omega^k\}$. Then

$$d\begin{pmatrix}\omega^3\\ \theta^3_1\\ \theta^3_2\end{pmatrix} \equiv \begin{pmatrix} 0 & 0\\ 2h_{21}\omega^2_1 - dh_{11} & (h_{22}-h_{11})\omega^2_1 - dh_{12}\\ (h_{22}-h_{11})\omega^2_1 - dh_{12} & -2h_{21}\omega^2_1 - dh_{22}\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \operatorname{mod} I$$
$$\equiv \begin{pmatrix}\pi^0_1 & \pi^0_2\\ \pi^1_1 & \pi^1_2\\ \pi^2_1 & \pi^2_2\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \operatorname{mod} I,$$

where the π^a_i's stand for the components of the matrix on the first line. The symbol relations are

$$\pi^0_1 \equiv \pi^0_2 \equiv \pi^2_1 - \pi^1_2 \equiv 0 \operatorname{mod} I.$$

There is no torsion, and $s_1 = 2, s_2 = 1$. Integral elements are given by

$$\begin{aligned}
\pi^0_1 &= \pi^0_2 = 0,\\
\pi^1_1 &= a\omega^1 + b\omega^2,\\
\pi^1_2 &= \pi^2_1 = b\omega^1 + c\omega^2,\\
\pi^2_2 &= c\omega^1 + e\omega^2,
\end{aligned}$$

where a, b, c, e are arbitrary. Thus $\dim A^{(1)} = 4 = s_1 + 2s_2$, and the system is involutive. The Cartan-Kähler Theorem implies that integral manifolds depend on one function of two variables. This should come as no surprise, since locally any surface can be written as a graph $z = f(x, y)$.

Example 5.8.2 (Cartan III: Linear Weingarten surfaces). Let A, B, C be constants and let $AK + 2BH + C = 0$ be a linear relation between functions H and K. We set up and describe the space of general integral manifolds for first-order adapted lifts of surfaces in $\mathbb{R}^3$, together with the components of the second fundamental form, having Gauss and mean curvatures satisfying this relation. (Note that this class of examples includes constant mean curvature and constant Gauss curvature surfaces.)

This is the system above restricted to a codimension one submanifold $\Sigma \subset ASO(3) \times \mathbb{R}^3{}_{h_{ij}}$. The submanifold is defined by the equation

$$A(h_{11}h_{22} - h_{12}^2) + B(h_{11} + h_{22}) + C = 0.$$

The differential of this equation implies that on Σ we may write

$$dh_{12} = \alpha dh_{11} + \beta dh_{22}$$

for some functions α, β, and thus

$$\pi^1_2 = \alpha\pi^1_1 + \beta\pi^2_2.$$

We still have no torsion and now have characters $s_1 = 2, s_2 = 0$. By Exercise 4.5.9.1 such a tableau is involutive, and solutions depend locally on two

functions of one variable. This suggests that an appropriate Cauchy problem for determining such surfaces would be to start out with a space curve (the two functions being the curvature and torsion of the curve) and then solve for a unique surface containing the curve. This construction is carried out explicitly in [**31**].

Exercise 5.8.3: Set up and perform Cartan's Test for the systems analogous to the above two examples, replacing $\mathbb{R}^3$ by a space form $X^3(\epsilon)$ (see §2.7 for frames for space forms).

Example 5.8.4 (Cartan II). Let $(S,g),(\underline{S},\underline{g})$, be surfaces with analytic Riemannian metrics. We set up and solve the EDS for maps $\phi : S \to \underline{S}$ that preserve the metric up to scale, i.e., for conformal maps from S to $\underline{S}$.

Let $\mathcal{F}(S)$ denote the orthonormal coframe bundle of S, with coframing $\omega^1, \omega^2, \omega^2_1 = -\omega^1_2$ satisfying $d\omega^i = -\omega^i_j\omega^j$. Similarly let $\mathcal{F}(\underline{S})$ denote the orthonormal coframe bundle of $\underline{S}$ with coframing $\underline{\omega}^1, \underline{\omega}^2, \underline{\omega}^2_1$. The metric on S is given by the pullback of $(\omega^1)^2 + (\omega^2)^2$ from any section $s : S \to \mathcal{F}(S)$, and that on $\underline{S}$ by the pullback of $(\underline{\omega}^1)^2 + (\underline{\omega}^2)^2$. Thus a conformal map $\phi : S \to \underline{S}$ induces a map $\Phi : \mathcal{F}(S) \to \mathcal{F}(\underline{S})$ such that $\Phi^*((\underline{\omega}^1)^2 + (\underline{\omega}^2)^2) = \lambda^2((\omega^1)^2 + (\omega^2)^2)$ for some positive function λ.

Without loss of generality, we may require that $\Phi^*(\underline{\omega}^j) = \lambda\omega^j$ for some function λ, because we have freedom to rotate in the tangent spaces at each point. So on $\Sigma := \mathcal{F}(S) \times \mathcal{F}(\underline{S}) \times \mathbb{R}_\lambda$, $\lambda > 0$, let

$$\begin{aligned} I &= \{\theta^1 := \omega^1 - \lambda\underline{\omega}^1,\ \theta^2 := \omega^2 - \lambda\underline{\omega}^2\},\\ J &= \{\theta^1, \theta^2, \omega^1, \omega^2, \omega^2_1\} = \{\theta^1, \theta^2, \underline{\omega}^1, \underline{\omega}^2, \underline{\omega}^2_1\}. \end{aligned}$$

We compute

$$\begin{aligned} d\theta^1 &= -\,\omega^1_2 \wedge \omega^2 - d\lambda \wedge \underline{\omega}^1 + \lambda\underline{\omega}^1_2 \wedge \underline{\omega}^2,\\ d\theta^2 &= -\,\underline{\omega}^2_1 \wedge \omega^1 - d\lambda \wedge \underline{\omega}^2 + \lambda\underline{\omega}^2_1 \wedge \underline{\omega}^1. \end{aligned}$$

Reducing mod I, we obtain

$$d\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} \equiv \begin{pmatrix} -\frac{d\lambda}{\lambda} & -(\underline{\omega}^2_1 - \omega^2_1) & 0\\ \underline{\omega}^2_1 - \omega^2_1 & -\frac{d\lambda}{\lambda} & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^2_1\end{pmatrix}.$$

Since $-\frac{d\lambda}{\lambda}$ and $\underline{\omega}^2_1 - \omega^2_1$ are independent mod J, there is no torsion, and we may write

$$A = \left\{ \begin{pmatrix} a & -b & 0\\ b & a & 0\end{pmatrix} \middle|\, a, b \in \mathbb{R} \right\}.$$

So $s_1 = 2, s_2 = 0$. By Definition 4.5.4, the tableau is involutive, and solutions depend on two functions of one variable.

We may explicitly realize the dependence of conformal maps between surfaces on two functions of one variable as follows:

Specify parametrized curves $f : \mathbb{R} \to S$, $g : \mathbb{R} \to \underline{S}$ (our two functions of one variable) and look for conformal maps ϕ such that $\phi(f(t)) = g(t)$, where t is the coordinate on $\mathbb{R}$. The claim is that up to constants there is a unique such map. In fact, choose an orthonormal frame e_1, e_2 on S such that e_1 is tangent to $f(\mathbb{R})$, and adapt similarly on $\underline{S}$. Set $\lambda = \frac{g'(t)}{f'(t)}$; then the map is uniquely determined up to constants.

Conformal maps are exactly the injective holomorphic or antiholomorphic maps between surfaces. These are determined by their restriction to an arc, so picking two parametrized arcs determines a holomorphic or antiholomorphic map. Even without knowing the coincidence of Lie groups $GL(1,\mathbb{C}) \simeq CO(2)$, one might have guessed this by observing that the tableau A is the same as the tableau for the Cauchy-Riemann equations (see Chapter 4, Appendix C) augmented with zeros.

Corollary 5.8.5 (Existence of isothermal coordinates). *Let S be a surface with an analytic Riemannian metric g. Let $p \in S$. Then there exist local coordinates (x,y) centered at p, such that $g = \lambda^2(dx^2 + dy^2)$, $\lambda \neq 0$, in some neighborhood of p. Such coordinates are called isothermal coordinates.*

Proof. Take $\underline{S} = \mathbb{R}^2$ with its flat metric. □

Example 5.8.6 (Lagrangian 3-manifolds in $\mathbb{C}^3$). Let $(\mathbb{R}^6, \omega, \langle\, ,\rangle)$ have its standard inner product and symplectic form

$$\omega = dx^1 \wedge dx^4 + dx^2 \wedge dx^5 + dx^3 \wedge dx^6.$$

We may also consider it as $\mathbb{C}^3$ with its standard Hermitian structure (see Appendix C). We will set up and solve an EDS for three-manifolds whose tangent spaces are isotropic for ω. This will amount to the same system as just looking for Lagrangian submanifolds of $\mathbb{R}^6$ just equipped with the symplectic form, but studying the problem in this way will help to serve as a warmup for studying special Lagrangian submanifolds. It is well-known in symplectic geometry [**47**] that Lagrangian submanifolds are locally given by lifts of graphs of functions to the cotangent space, so it is not surprising that the integral manifolds will depend on one function of three variables.

Matrices in the Lie algebra $\mathfrak{u}(3)$ of $U(3) = SO(6) \cap Sp(\mathbb{R}^6, \omega)$ have the form

$$\begin{pmatrix} 0 & -\alpha_1^2 & -\alpha_1^3 & -\rho_1 & -\beta_1^2 & -\beta_1^3 \\ \alpha_1^2 & 0 & -\alpha_2^3 & -\beta_1^2 & -\rho_2 & -\beta_2^3 \\ \alpha_1^3 & \alpha_2^3 & 0 & -\beta_1^3 & -\beta_2^3 & -\rho_3 \\ \rho_1 & \beta_1^2 & \beta_1^3 & 0 & \alpha_1^2 & \alpha_1^3 \\ \beta_1^2 & \rho_2 & \beta_2^3 & -\alpha_1^2 & 0 & \alpha_2^3 \\ \beta_1^3 & \beta_2^3 & \rho_3 & -\alpha_1^3 & -\alpha_2^3 & 0 \end{pmatrix}.$$

Let $\mathcal{F}_{U(3)}$ be the unitary frame bundle for $\mathbb{C}^3$, and let $I = \{\omega^4, \omega^5, \omega^6\}$ be defined on $\mathcal{F}_{U(3)}$, with independence condition $\Omega = \omega^1 \wedge \omega^2 \wedge \omega^3$. We have structure equations

$$d(x, e_1, \ldots, e_6) = (x, e_1, \ldots, e_6) \begin{pmatrix} 0 & 0 & & \cdots & & & 0 \\ \omega^1 & 0 & -\alpha_1^2 & -\alpha_1^3 & -\rho_1 & -\beta_1^2 & -\beta_1^3 \\ \omega^2 & \alpha_1^2 & 0 & -\alpha_2^3 & -\beta_1^2 & -\rho_2 & -\beta_2^3 \\ \omega^3 & \alpha_1^3 & \alpha_2^3 & 0 & -\beta_1^3 & -\beta_2^3 & -\rho_3 \\ \omega^4 & \rho_1 & \beta_1^2 & \beta_1^3 & 0 & \alpha_1^2 & \alpha_1^3 \\ \omega^5 & \beta_1^2 & \rho_2 & \beta_2^3 & -\alpha_1^2 & 0 & \alpha_2^3 \\ \omega^6 & \beta_1^3 & \beta_2^3 & \rho_3 & -\alpha_1^3 & -\alpha_2^3 & 0 \end{pmatrix}.$$

The Maurer-Cartan equation (1.20) implies that

$$d \begin{pmatrix} \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} \equiv \begin{pmatrix} \rho_1 & \beta_1^2 & \beta_1^3 \\ \beta_1^2 & \rho_2 & \beta_2^3 \\ \beta_1^3 & \beta_2^3 & \rho_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \bmod I.$$

We see there is no torsion, and $s_1 = 3, s_2 = 2, s_3 = 1$. (Note that the tableau corresponds to the lower left hand block in $\mathfrak{u}(3)$.)

Exercise 5.8.7: Show that this tableau is involutive.

Proposition 5.8.8. *Analytic Lagrangian 3-folds in $\mathbb{C}^3$ depend locally on one function of 3 variables.*

Exercise 5.8.9: Do a similar computation for Lagrangian n-folds in $\mathbb{C}^n$ for arbitrary n (see also Example 7.5.5

Remark 5.8.10 (Another interpretation). The Euclidean inner product $\langle \, , \rangle$ and ω determine an almost complex structure J by $\omega(v, w) = \langle v, Jw \rangle$. The integral manifolds above can also be characterized as the three-manifolds such that $J(T_xM) \perp T_xM$.

Example 5.8.11 (Minimal surfaces in $\mathbb{E}^3$ with a specified curvature function). For simplicity, we assume no umbilic points and use Darboux framings. On $\mathcal{F}_{\mathbb{E}^3}$ let $K(x)$ be a given function which will be the curvature and let $k(x) = \sqrt{-K(x)}$ be the positive square root.

Let $I = \{\omega^3, \theta_1^3, \theta_2^3\}$ and let $J = \{\omega^3, \theta_1^3, \theta_2^3, \omega^1, \omega^2\}$, where we set $\theta_1^3 = \omega_1^3 - k\omega^1$ and $\theta_2^3 = \omega_2^3 + k\omega^2$. The structure equations are

$$d \begin{pmatrix} \omega^3 \\ \theta_1^3 \\ \theta_2^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ -dk & -2k\omega_1^2 \\ -2k\omega_1^2 & dk \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

Write $dk = k_1\omega^1 + k_2\omega^2$. The symbol relations are

$$\left.\begin{aligned}\pi_1^0 \equiv \pi_2^0 \equiv 0\\ \pi_1^2 - \pi_2^1 \equiv 0\\ \pi_1^1 + \pi_2^2 \equiv 0\\ \pi_1^1 \equiv k_1\omega^1 + k_2\omega^2\end{aligned}\right\} \operatorname{mod} I.$$

Again we can change bases to attempt to get rid of the apparent torsion. If we rewrite our equations as

$$d\begin{pmatrix}\omega^3\\ \theta_1^3\\ \theta_2^3\end{pmatrix} \equiv \begin{pmatrix}0 & 0\\ 0 & -2k\omega_1^2 + k_2\omega^1\\ -2k\omega_1^2 - k_1\omega^2 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},$$

the symbol relations become

$$\left.\begin{aligned}\pi_1^0 \equiv \pi_2^0 \equiv 0\\ \pi_1^2 - \pi_2^1 \equiv -k_1\omega^1 + k_2\omega^2\\ \pi_1^1 \equiv \pi_2^2 \equiv 0\end{aligned}\right\} \operatorname{mod} I.$$

This still has apparent torsion, but if we write instead

$$d\begin{pmatrix}\omega^3\\ \theta_1^3\\ \theta_2^3\end{pmatrix} \equiv \begin{pmatrix}0 & 0\\ 0 & -2k\omega_1^2 + k_2\omega^1 - k_1\omega^2\\ -2k\omega_1^2 + k_2\omega^1 - k_1\omega^2 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},$$

the symbol relations become

$$\pi_1^0 \equiv \pi_2^0 \equiv \pi_1^1 \equiv \pi_2^2 \equiv \pi_1^2 - \pi_2^1 \equiv 0 \operatorname{mod} I.$$

We see that $s_1 = 1, s_2 = 0$, and $\dim A^{(1)} = 0$, so the system is not involutive.

We can short-cut the prolongation process by realizing that, since the system contains both $\pi_2^1 \wedge \omega^1$ and $\pi_2^1 \wedge \omega^2$, then π_2^1 must vanish on all integral manifolds satisfying the independence condition. Thus, we need to add $\pi := \pi_2^1 = -2k\omega_1^2 + k_2\omega^1 - k_1\omega^2$ to the system. Call the new system I^+. By counting dimensions, we see that I^+ will either be Frobenius at x or have no integral manifold passing through x. We have $d\omega^3 \equiv d\theta_1^3 \equiv d\theta_2^3 \equiv 0 \operatorname{mod} I^+$, and we compute

$$\begin{aligned}d\pi &= d(-2k\omega_1^2 + k_2\omega^1 - k_1\omega^2)\\ &\equiv \left(-k_{1,1} - k_{2,2} + \frac{k_1^2 + k_2^2}{2k} + 2kK\right)\omega^1 \wedge \omega^2 \operatorname{mod} I^+,\end{aligned}$$

where we have written $dk_j = k_{j,1}\omega^1 + k_{j,2}\omega^2$. The integrability condition we have obtained can be put more simply:

Exercise 5.8.12: Recall the Laplacian on functions defined in §2.6. Show that

$$\Delta \log(-K) - 4K = \frac{2}{k}(k_{1,1} + k_{2,2} - \frac{k_1^2 + k_2^2}{2k} - 2kK).$$

In summary:

Theorem 5.8.13 (Ricci). *Let (M, g) be a surface with a Riemannian metric. Let $K(x)$ denote the curvature function. Away from umbilic points, necessary and sufficient conditions for M to be locally minimally and isometrically immersed in $\mathbb{R}^3$ are that $K < 0$ and that $\Delta \log(-K) = 4K$. If K is nonconstant, then there is a one-parameter family of such immersions, up to congruence.*

For more results along these lines, see [**38**].

Note that we did not need the Cartan-Kähler Theorem to prove this theorem, since it reduced to the Frobenius Theorem. But the integrability conditions were more easily obtained by following the Cartan algorithm.

Example 5.8.14 (Cartan IV: existence of "curvature-line coordinates", i.e., isothermal coordinates along lines of curvature). Given $M^2 \subset \mathbb{E}^3$, we ask if there exist isothermal coordinates (x, y) on M such that the principal directions are $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. In this case the curves where either x or y is constant are such that their tangent directions are principal directions. Curves whose tangent directions are principal directions are (somewhat misleadingly) called *lines of curvature.*

An EDS whose integral manifolds correspond to surfaces in $\mathbb{E}^3$ together with isothermal coordinates along lines of curvature may be defined on $ASO(3) \times \mathbb{R}^2_{k_1,k_2} \times \mathbb{R}^1_u$ by

$$\mathcal{I} = \{\omega^3, \omega^3_1 - k_1\omega^1, \omega^3_2 - k_2\omega^2, d(u\omega^1), d(u\omega^2)\},$$
$$\Omega = \omega^1 \wedge \omega^2.$$

Here, the forms $\omega^3_1 - k_1\omega^1, \omega^3_2 - k_2\omega^2$ are in the ideal to ensure that $(\omega^1)^\perp$, $(\omega^2)^\perp$ correspond to principal directions on the surface, and k_1, k_2 will be the principal curvatures. If we can multiply ω^1, ω^2 by a common function u to make them both closed, then the fact that closed forms are locally exact implies local existence of the isothermal coordinates.

Note that $\mathcal{I}$ is not a Pfaffian system. But if we adjoin new variables u_1, u_2 such that $du = u_1\omega^1 + u_2\omega^2$, then the generators of $\mathcal{I}$ of degree two are

$$d(u\omega^1) = (-u_2\omega^1 + \omega^2_1) \wedge \omega^2,$$
$$d(u\omega^2) = (-u_1\omega^2 - \omega^2_1) \wedge \omega^1.$$

Thus, an equivalent system is (I, Ω) defined on $ASO(3) \times \mathbb{R}^2_{k_1,k_2} \times \mathbb{R}^1_u \times \mathbb{R}^2_{u_1,u_2}$ with

$$I := \{\omega^3, \omega^3_1 - k_1\omega^1, \omega^3_2 - k_2\omega^2, du - u_1\omega^1 - u_2\omega^2, \omega^2_1 - u_2\omega^1 + u_1\omega^2\},$$

which is a linear Pfaffian system.

Exercise 5.8.15: Determine the conditions on k_1, k_2 such that there exist integral manifolds of the above system. (The answer will be in the form of a PDE that must be satisfied in addition to the Codazzi equation.) Determine the space of integral manifolds for generic functions k_1, k_2 that satisfy these conditions.

Exercises 5.8.16:
1. **(Cartan VI)** Set up an EDS for pairs of surfaces $S, S' \subset \mathbb{E}^3$ together with a local diffeomorphism $\phi : S \to S'$ that is an isometry preserving a family of asymptotic directions. What does the general solution depend on? Geometrically characterize the surfaces S admitting a non-congruent mate S'.
2. **(Cartan VII)** Set up an EDS for pairs of surfaces $S, S' \subset \mathbb{E}^3$ together with a local diffeomorphism $\phi : S \to S'$ that is an isometry preserving a family of lines of curvature. What does the general solution depend on? Geometrically characterize the surfaces S admitting a non-congruent mate S'.
3. **(Cartan XII)** Set up an EDS for pairs of surfaces $S, S' \subset \mathbb{E}^3$ such that there exists a diffeomorphism $\phi : S \to S'$ that is a conformal map preserving both families of asymptotic directions. What does the general solution depend on? Show that, if S is not minimal, there exists at most one such S', up to congruence. In the special case where S is minimal, what are the surfaces S' such that there exists such a ϕ?

5.9. Functions whose Hessians commute, with remarks on singular solutions

We set up an exterior differential system for s functions $f^a : \mathbb{R}^n \to \mathbb{R}$, $1 \leq a, b \leq s$, whose Hessians commute at every point. This system was motivated by a problem in variation of Hodge structures; see [**25, 118**].

Let $J^2(\mathbb{R}^n, \mathbb{R}^s)$ have coordinates $(x^i, u^a, p^a_i, p^a_{ij})$, where $1 \leq i, j \leq n$, with the canonical contact system generated by $\theta^a = du^a - p^a_i dx^i$ and $\theta^a_i = dp^a_i - p^a_{ij}dx^j$.

Let $\Sigma \subset J^2(\mathbb{R}^n, \mathbb{R}^s)$ be the subset defined by $[p^a, p^b] = 0$, where $p^a = (p^a_{ij})$ and $[,]$ is the commutator of matrices. At each $f \in \Sigma$ there exists a matrix $h \in O(n)$ and eigenvalues l^a_i such that each $p^a = hl^ah^{-1}$ for a diagonal matrix $(l^a)_{ij} = l^a_i\delta_{ij}$. We restrict attention to the open subset U of

Σ on which the common eigenspaces of the p^a are all one-dimensional. Then h is unique up to permutations, and h and the l^a_i are smooth functions.

Functions with commuting Hessians are in one-to-one correspondence with integral manifolds of the restriction I of the contact system on $J^2(\mathbb{R}^n, \mathbb{R}^s)$ to Σ. We will write the second set of generators for I in vector form as $\Theta^a = d(p^a) - hl^a h^{-1}dx$. Let $\omega^i = (h^{-1}dx)^i$ and $\phi^i_j = (h^{-1}dh)^i_j$. We will use matrix notation; in particular, the skew-symmetric matrix $\phi = (\phi^i_j)$ satisfies $d\omega = -\phi \wedge \omega$ and the Maurer-Cartan equation $d\phi = -\phi \wedge \phi$.

To perform Cartan's Test, we calculate the exterior derivatives of the forms in I. We get $d\theta^a \equiv 0 \mod I$ and

$$d\Theta^a \equiv -h([\phi, l^a] + dl^a) \wedge \omega \mod I.$$

Since the matrix h is invertible, we can ignore it when computing characters and $\dim A^{(1)}$. (Our choice of notation has the unfortunate consequence that $\dim W = ns$ for this tableau.)

Let $\psi^a = [l^a, \phi]$, so $\psi^{ai}_j = (l^a_j - l^a_i)\phi^i_j$, and in particular $\psi^{ai}_i = 0$ and $\psi^{ai}_j = \psi^{aj}_i$ for $i \neq j$. There is no torsion, and the tableau consists of s blocks:

$$\begin{pmatrix} dl^a_1 & \psi^{a1}_2 & \cdots & \psi^{a1}_n \\ \psi^{a2}_1 & dl^a_2 & \cdots & \psi^{a2}_n \\ \vdots & & & \\ \psi^{an}_1 & \cdots & \psi^{an}_{n-1} & dl^a_n \end{pmatrix}.$$

Because all the dl^a_i's are independent, we see that $s_1 = ns$ (e.g., by adding the columns together). There are $\binom{n}{2}$ remaining independent entries in the tableau. For generic values of the eigenvalues, we can recover the maximum number of independent entries in each column. Thus

$$s_2 = \min\{sn, \tbinom{n}{2}\},\ s_3 = \min\{sn, \tbinom{n}{2} - sn, 0\},\ s_4 = \min\{sn, \tbinom{n}{2} - 2sn, 0\},$$

etc. To calculate $\dim A^{(1)}$, suppose

$$dl^1_j = a_{jk}\omega^k, \tag{5.19}$$

which is a choice of n^2 constants. Then

$$\psi^{1i}_j = a_{ij}\omega^i + a_{ji}\omega^j + \sum_{k \neq i,j} b^i_{jk}\omega^k,$$

where the a_{ij}'s are from (5.19) and the b^i_{jk}'s are additional constants which must satisfy $b^i_{jk} = b^i_{kj} = b^j_{ik}$.

Let $2 \leq \rho, \sigma \leq s$ and let $m^{\rho i}_j = (l^\rho_i - l^\rho_j)/(l^1_i - l^1_j)$. Then $\psi^{\rho i}_j = m^{\rho i}_j \psi^{1i}_j$ for each $i \neq j$, and consequently $m^{\rho i}_j b^i_{jk} = m^{\rho i}_k b^i_{kj}$. Since generically the values of the $m^{\rho i}_j$ are all distinct, all the b^i_{jk} must be zero. The only remaining

freedom in choosing an integral element are the "diagonal" elements, i.e., we may set

$$dl^\rho_j = m^{\rho i}_{\ j} a_{ij}\omega^i + c^\rho_j \omega^j,$$

where the c^ρ_j's are independent constants. Thus

$$\dim A^{(1)} = n^2 + (s-1)n = 2\binom{n}{2} + s_1.$$

Thus if we are in the case where $\binom{n}{2} \leq sn$, i.e., $s \geq \frac{n-1}{2}$, then the system is involutive and general integral manifolds depend on $\binom{n}{2}$ functions of two variables and sn functions of one variable.

There are many other families of integral manifolds to this system. For example: a naïve way to construct s functions of n variables whose Hessians commute is to take the first function to be arbitrary, and the rest of the form $f^a(x^1, \ldots, x^n) = u^a(x^1 + \ldots + x^n)$, so that the Hessians of f^a are all multiples of the identity matrix. Such integral manifolds are called *singular solutions.*

From the perspective of EDS, these integral manifolds are obtained by restricting to the stratum $\Sigma' \subset \Sigma$ where $p^a_{ij} = C^a \delta_{ij}$ for $a > 1$. Note that they depend roughly on one function of n variables, and thus the space of these integral manifolds is larger than for integral manifolds with generic initial data.

Singular solutions. So far we have discussed integral manifolds of a system (I, J) on Σ obtained by solving a series of Cauchy problems starting at a generic point $x \in \Sigma$ and a generic flag in $T_x\Sigma$ among flags leading to n-dimensional integral elements. A *singular solution* to an exterior differential system is an integral manifold obtained by either starting at a non-generic point $y \in \Sigma$, or taking a non-generic flag for the Cauchy problems.

If one is at a non-generic point $y \in \Sigma$ and restricts to the submanifold $\Sigma' \subset \Sigma$ of points near y having the same non-genericity properties, one can again use the Cartan algorithm restricted to Σ' to find integral manifolds.

For singular solutions corresponding to a non-generic flag, one often can still apply Cartan's algorithm to a modified system. In small dimensions, singular solutions can often be treated on a case by case basis; see the numerous examples in [**31**].

5.10. The Cartan-Janet Isometric Embedding Theorem

We return to the isometric embedding problem. Following the proof in [**10**], we will show that $r(n) = \binom{n}{2}$.

Theorem 5.10.1 (Cartan-Janet). *Let (M^n, g) be an analytic Riemannian manifold. Then there exist local isometric embeddings of M^n into $\mathbb{E}^{n+\binom{n}{2}}$ that depend on n functions of $n-1$ variables.*

We first need to see that the variety $\Sigma' \subset \Sigma$ defined by (5.6) is nonempty, i.e., that no matter what the curvature tensor of M is at a point x, there is some second fundamental form giving rise to that curvature tensor. As in 5.4.5, we let $V = \mathbb{R}^n$, $W = (\mathbb{R}^s, \langle\,,\rangle)$ and define

$$\gamma : S^2V^* \otimes W \to \mathcal{K}(V),$$
$$h^a_{ij} v^i v^j \otimes w_a \mapsto \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}) v^i \otimes v^j \otimes v^k \otimes v^l, \tag{5.20}$$

where $\mathcal{K}(V) = S_{22}(V)$ is the kernel of the skew-symmetrization map $\delta : \Lambda^2 V \otimes \Lambda^2 V \to \Lambda^4 V$. Let δ' denote the restriction to $\mathcal{K}(V) \otimes V$ of the natural map

$$\mathrm{Id} \otimes \delta : \Lambda^2 V \otimes (\Lambda^2 V \otimes V) \to \Lambda^2 V \otimes \Lambda^3 V$$

given by skew-symmetrization on the last two factors, and define $\mathcal{K}^{(1)}(V) = \ker \delta'$. We saw in Appendix A that $d\Theta_x \in \mathcal{K}^{(1)}(T^*_x M)$, i.e., that $\mathcal{K}^{(1)}(V)$ is the space of derivatives of the curvature tensor.

Lemma 5.10.2. *If $s \geq \binom{n}{2}$, then γ is surjective.*

Proof. Let $V_- = \{v^1, \dots, v^{n-1}\}$ and let $1 \leq \alpha, \beta, \gamma, \epsilon \leq n-1$.

Assume the lemma is true for $W \otimes S^2 V_- \to \mathcal{K}(V_-)$; we will show it is true for $W \otimes S^2 V \to \mathcal{K}(V)$. (Note that $\mathcal{K}(\mathbb{R}^1) = 0$.)

Let $R \in \mathcal{K}(V)$. By assumption, there are vectors $h_{\alpha\beta} = h^a_{\alpha\beta} w_a \in W$ such that

$$R_{\epsilon\alpha\beta\gamma} = h_{\alpha\gamma} \cdot h_{\epsilon\beta} - h_{\alpha\beta} \cdot h_{\epsilon\gamma},$$

where $h_{\alpha\gamma} \cdot h_{\epsilon\beta} := \sum_a h^a_{\alpha\gamma} h^a_{\epsilon\beta}$. Let $W_- = \{h_{\alpha\beta}\}$. It is sufficient to find new vectors $w_0, w_1, \dots, w_{n-1} \in W \backslash W_-$ such that

$$h_{\alpha\gamma} \cdot w_\beta - h_{\alpha\beta} \cdot w_\gamma = R_{n\alpha\beta\gamma},$$
$$h_{\alpha\beta} \cdot w_0 - w_\alpha \cdot w_\beta = R_{n\alpha n\beta},$$

because if they exist, we may take

$$h_{\alpha n} = w_\alpha,$$
$$h_{nn} = w_0,$$

to obtain $\gamma(h) = R$. To find appropriate vectors w_0, w_α, consider the following:

Write $R' = R_{n\alpha\beta\gamma} v^\alpha \otimes (v^\beta \wedge v^\gamma) \in V_- \otimes \Lambda^2 V_-$. Note that $R' \in S_{21}(V_-)$, because it is in the kernel of the map $\delta : V_- \otimes \Lambda^2 V_- \to \Lambda^3 V_-$. Recall from Appendix A that $S_{21}(V_-)$ may also be described as the image of the natural map $q : S^2 V_- \otimes V_- \to V_- \otimes \Lambda^2 V_-$. Thus, we may write $R' = q(r)$ for some

$r \in S^2V_- \otimes V_-$. In indices, $R_{n\alpha\beta\gamma} = r_{\alpha\beta\gamma} - r_{\alpha\gamma\beta}$. We then seek vectors w_α such that $h_{\alpha\beta} \cdot w_\gamma = r_{\alpha\beta\gamma}$ and $h_{\alpha\beta}, w_\gamma$ are linearly independent. Since the inner product on W is nondegenerate and $\dim W - \dim W_- \geq n$, we can always find such w_α. Similarly, once we have done so we can find w_0 such that $h_{\alpha\beta} \cdot w_0 = R_{n\alpha n\beta} + w_\alpha \cdot w_\beta$, proving surjectivity. $\square$

Proof of Cartan-Janet. Our system is $(\tilde{I}, \tilde{J})$ of (5.5), restricted to the submanifold Σ' given by (5.6). We set $r = \binom{n}{2}$. We need to show that there is no torsion and that the tableau is involutive.

Fix a general point $y = ((x,u),(p,v),h) \in \Sigma' \subset \mathcal{F}_M \times \mathcal{F}_{\mathbb{E}^{n+r}} \times W \otimes S^2V^*$. In particular, $h \in W \otimes S^2V^*$ satisfies the Gauss equations and, since γ is a smooth map at general points and h is a general point, $\gamma_{*h} : T_h(W \otimes S^2V^*) \to T_{\gamma(h)}\mathcal{K}(V)$ is defined and surjective. We form a new map by tensoring γ_{*h} with V^*. We identify $T_h(W \otimes S^2V^*)$ with $W \otimes S^2V^*$, $T_{\gamma(h)}\mathcal{K}(V)$ with $\mathcal{K}(V)$, and symmetrize both sides to obtain a linear map

$$\tilde{\gamma}_{*,h} : W \otimes S^3V^* \to \mathcal{K}^{(1)}(V^*) \tag{5.21}$$

which is also surjective.

To show the apparent torsion is absorbable, we first show that it lives in $\mathcal{K}^{(1)}(V)$. Note that integral elements at y are given by the equations $\gamma(h, dh) = dR$. We may also write these equations as

$$\gamma(h, \pi) = \nabla R.$$

where $\pi = \pi^\mu_{jk} w_\mu v^j v^k$, and we define

$$\pi^\mu_{jk} := -dh^\mu_{jk} - h^\nu_{jk}\omega^\mu_\nu + h^\mu_{jl}\omega^l_k + h^\mu_{kl}\omega^l_j$$

and

$$(\nabla R)_{ijkl} = dR_{ijkl} + \gamma(h_{il}, (h^\nu_{jk}\omega^\mu_\nu + h^\mu_{jm}\omega^l_k + h^\mu_{km}\omega^m_j)w_\mu).$$

Since γ is quadratic, $\gamma_{*,h}(\pi) = \gamma(h, \pi)$, Thus, $[T] = 0$ by the surjectivity of $\tilde{\gamma}_{*h}$.

We now need to calculate $\dim A^{(1)}$ and the characters s_k. To show the system is involutive, it is sufficient to show $\dim A^{(1)} \geq s_1 + 2s_2 + \ldots + ns_n$. We will estimate both sides to obtain the inequality (which of course then must be an equality).

On an integral element we have $\pi^a_{ij} = h^a_{ijk}\eta^k$ for some constants h^a_{ijk}, and $\dim A^{(1)}$ is the dimension of the space of choices of such constants. The surjectivity of (5.21) implies

$$\begin{aligned} \dim A^{(1)} &\geq \dim W \otimes S^3V^* - \dim \mathcal{K}^{(1)}(V) \\ &= r\binom{n+2}{3} - \frac{n^2(n^2-1)(n+2)}{24}. \end{aligned} \tag{5.22}$$

We now estimate the characters. Fix $p < n$ and let $1 \leq s, t, u \leq p-1$, $p \leq x, y, z \leq n$. Consider the equations

$$h_{tu} \cdot \pi_{px} = -h_{px} \cdot \pi_{tu} + h_{tx} \cdot \pi_{up} + h_{pu} \cdot \pi_{tx}, \tag{5.23}$$

$$h_{tx} \cdot \pi_{py} - h_{ty} \cdot \pi_{px} = -h_{py} \cdot \pi_{tx} + h_{px} \cdot \pi_{ty}. \tag{5.24}$$

Equation (5.23) provides $\binom{p}{2}(n-p+1)$ equations on the π_{px} in terms of the π_{sj}. (The first factor is the number of choices of pairs $t \leq u$, the second the number of choices for x.) Equation (5.24) provides $(p-1)\binom{n-p+1}{2}$ equations. Thus

$$s_p \leq r(n-p+1) - \binom{p}{2}(n-p+1) - (p-1)\binom{n-p+1}{2}$$

and $s_n = 0$. Therefore

$$\begin{aligned} &s_1 + 2s_2 + \ldots + ns_n \\ &\qquad \leq \sum_{p=1}^{n-1} p\left[r(n-p+1) - \binom{p}{2}(n-p+1) - (p-1)\binom{n-p+1}{2}\right]. \end{aligned}$$

The summation on the right-hand side equals (5.22), so we obtain the desired inequality. □

5.11. Isometric embeddings of space forms (mostly flat ones)

We have seen that an arbitrary analytic Riemannian n-fold locally isometrically embeds into $\mathbb{E}^{n+\binom{n}{2}}$. However, special metrics might embed into smaller Euclidean spaces. For example, the round sphere occurs as a hypersurface. Here we study embeddings of the other two types of space forms, flat metrics and hyperbolic metrics, concentrating on the case of flat metrics.

We have seen any flat surface in $\mathbb{E}^3$ must have a degenerate Gauss map. Here we only study flat submanifolds of $\mathbb{E}^{n+r}$ with immersive Gauss mappings. This is equivalent to saying that the second fundamental form II is such that for all $v \in T_xM$, there exists $w \in T_xM$ such that $II(v,w) \neq 0$. We call such forms *nondegenerate*.

Lemma 5.11.1 (Cartan). *Let $V = \mathbb{R}^n$, $W = \mathbb{R}^r$, and let W have a nondegenerate inner product $\langle\, ,\rangle$. Let $h \in W \otimes S^2V^*$ be such that $\gamma(h) = 0$ and h is nondegenerate. Then $r \geq n$, and if $r = n$ there exist a basis $v^1, \ldots, v^n$ of V^* and an orthonormal basis $w_1, \ldots, w_r$ of W such that $h = v^i \circ v^i \otimes w_i$.*

Corollary 5.11.2 (Cartan). *There are no local isometric embeddings of a flat Riemannian manifold (M^n, g) into $\mathbb{E}^{n+r}$ with nondegenerate second fundamental form for $r < n$.*

Proof. Assume h is as in the hypotheses of the lemma. Let $v_0 \in V$. Consider the linear map

$$v_0 \lrcorner h : V \to W,$$
$$v \mapsto h(v_0, v).$$

We claim that if v_0 is generic, then $v_0 \lrcorner h$ is injective, which will prove the first assertion. Let $1 \le s \le r$ and choose a basis $w_1, \dots, w_r$ of W so that $h = h^s w_s$ with $h^s \in S^2V^*$.

Say the claim were false, then consider $U_0 := \{v_0 \lrcorner h^s\} \subsetneq V^*$. Let $1 \le \xi, \eta \le \dim U_0$ and $\dim U_0 + 1 \le \phi, \psi \le r$. Adapt our basis of W so that $v_0 \lrcorner h^\psi = 0$ for all ψ and the $v_0 \lrcorner h^\xi$ span U_0. Let $v^\xi = v_0 \lrcorner h^\xi$. By hypothesis, for all $u, v \in V$, we have

$$0 = (u \wedge v) \lrcorner \gamma(h) = \sum_s (u \lrcorner h^s) \wedge (v \lrcorner h^s).$$

Consider

$$0 = (v \wedge v_0) \lrcorner \gamma(h) = \sum_\xi (v \lrcorner h^\xi) \wedge v^\xi.$$

By the Cartan Lemma A.1.9 we have

$$\{v \lrcorner h^\xi \mid v \in V, 1 \le \xi \le \dim U_0\} \subseteq U_0. \tag{5.25}$$

This implies $\mathrm{Singloc}\{h^\xi\} = \{v \lrcorner h^\xi\} \supseteq U_0{}^\perp$ and thus is nonempty. Let $v_1 \in \mathrm{Singloc}\{h^\xi\}$. There is some index ϕ and $v_2 \in V$ such that $h^\phi(v_1, v_2) \neq 0$, because $\mathrm{Singloc}(h) = 0$. For all $v \in V$, $v_0 + \epsilon v$ will be generic if ϵ is sufficiently small, because v_0 is generic. Consider

$$U_1 = \{(v_0 + \epsilon v_2) \lrcorner h^s\} = \{(v_0 + \epsilon v_2) \lrcorner h^\xi, \epsilon v_2 \lrcorner h^\phi\}.$$

By (5.25), $\{(v_0 + \epsilon v_2) \lrcorner h^\xi\} \subseteq U_0$, and in fact we have equality because $v_0 + \epsilon v_2$ is generic. But $\{\epsilon v_2 \lrcorner h^\phi\} \not\subset U_0$, which implies $\dim U_1 > \dim U_0$, giving a contradiction.

To prove the normal form, consider a basis $v_1, \dots, v_n$ of V where v_1 is h-generic. Let $2 \le \rho, \sigma \le n$. Again, since

$$(v_\rho \wedge v_1) \lrcorner \gamma(h) = 0,$$

the Cartan Lemma implies that $v_\rho \lrcorner h^s = C^s_t(\rho)(v_1 \lrcorner h^t)$ for some symmetric matrices $C(\rho)$. Using $(v_\rho \wedge v_\sigma) \lrcorner \gamma(h) = 0$, we see that the matrices commute, and are therefore simultaneously diagonalizable by an orthogonal matrix, say $B \in O(W)$. Let $q^s = B^s_t h^t$. We have arranged things so that for each index s, $\dim\{v_j \lrcorner q^s \mid 1 \le j \le n\} = 1$, so $\mathrm{rank}\, q^s = 1$ for all s. $\square$

We now study existence when $r = n$. (We follow the arguments in [**10**].) We have seen that if we are willing to abandon orthonormal frames, there is a nice normalization of the second fundamental form. In fact we don't need to abandon orthonormal frames entirely, just in the tangent space of M.

Let F be the subbundle of $\mathcal{F}_{\mathsf{GL}(\mathbb{E}^{2n})}$ consisting of frames $(x, e_1, \ldots, e_{2n})$ adapted so that

$$\begin{aligned}\langle e_i, e_{n+j}\rangle &= 0,\\ \langle e_{n+i}, e_{n+j}\rangle &= \delta_{ij}.\end{aligned}$$

If we set

$$G_{ij} = \langle e_i, e_j\rangle,$$

then the pullback of the Maurer-Cartan form of $\mathcal{F}_{\mathsf{GL}(\mathbb{E}^{2n})}$ to F has the relations

$$\omega^{n+i}_{n+j} + \omega^{n+j}_{n+i} = 0, \tag{5.26}$$

$$\omega^{n+i}_j + G_{jk}\omega^i_{n+i} = 0, \tag{5.27}$$

while otherwise the forms are independent. (Equation (5.27) is obtained from $0 = d\langle e_j, e_{n+j}\rangle$.)

Consider the Pfaffian system

$$I = \{\omega^{n+j}, \omega^{n+i}_j - \delta^{ij}\omega^j\}$$

with independence condition $\Omega = \omega^1 \wedge \ldots \wedge \omega^n$. Its integral manifolds are lifts of n-folds whose second fundamental form is in the normal form of the conclusion of Lemma 5.11.1. Thus the integral manifolds are flat, and all flat n-folds in $\mathbb{E}^{2n}$ with immersive Gauss maps occur as integral manifolds of I. We calculate

$$\begin{aligned}d\omega^{n+j} &\equiv 0 \operatorname{mod} I,\\ d(\omega^{n+i}_j - \delta^{ij}\omega^j) &\equiv \omega^i_j \wedge \omega^i - \omega^{n+i}_{n+j} \wedge \omega^k - \delta^{ij}\omega^i_l \wedge \omega^l \operatorname{mod} I.\end{aligned}$$

There is no torsion (as the only symbol relations are (5.26)), so we calculate the characters of the tableau A. We see that $s_1 = n^2$ and $s_2 = \binom{n}{2}$. (If this is not clear, try writing out the $n = 3$ case.) Since $\dim A$ is the number of independent forms among the $\omega^i_j, \omega^{n+i}_{n+j}$, we see $\dim A = n^2 + \binom{n}{2}$ and the remaining characters are zero. We show A is involutive by explicitly describing the space of integral elements: let $\{B^i_j \mid 1 \leq i, j \leq n\}$ and $\{C^i_j \mid i \neq j\}$ be constants, and set

$$\begin{aligned}\omega^{n+i}_{n+j} &= C^i_j\omega^i - C^j_i\omega^j,\\ \omega^i_i &= \sum_k B^i_k\omega^k,\\ \omega^i_j &= 2B^i_j\omega^i - C^i_j\omega^j.\end{aligned}$$

It is easy to verify that this determines an integral element, so $\dim A^{(1)} \geq n^2 + 2\binom{n}{2}$. But we already know that $\dim A^{(1)} \leq n^2 + 2\binom{n}{2}$, so we have equality, and the tableau is involutive.

In summary, we have shown:

Theorem 5.11.3 (Cartan [**28**]). *Local isometric embeddings $\mathbb{E}^n \to \mathbb{E}^{2n}$ with immersive Gauss maps exist and depend on $\binom{n}{2}$ functions of 2 variables.*

Local isometric embeddings of hyperbolic space. We have seen that hyperbolic 2-space H^2 can be locally embedded into $\mathbb{E}^3$. We now study H^n for $n > 2$.

Let g_0 denote the metric on H^n. Consider $\gamma^0 : S^2V^* \to \mathcal{K}$ obtained by taking $W = \mathbb{R}$ in (5.20). The Riemann curvature tensor of H^n is given by

$$R = -\gamma^0(g_0, g_0).$$

Thus in order to obtain isometric embeddings of H^n, we need second fundamental forms h satisfying $\gamma(h,h) = -\gamma^0(g_0, g_0)$. Here is a nifty trick due to Cartan: Let $\hat{W} = W \oplus \mathbb{R}\{e_0\}$, and $\hat{h} = h + g_0 e_0$. The admissible second fundamental forms h are exactly those such that $\gamma(\hat{h}, \hat{h}) = 0$. The hypotheses of Lemma 5.11.1 are satisfied for $\hat{h}$, and we conclude

Theorem 5.11.4 (Cartan [**28**]). *Hyperbolic space H^n does not admit any local isometric embeddings into $\mathbb{E}^{n+r}$ for $r < n-1$.*

There do exist local isometric embeddings for $r = n-1$; see [**28**] or [**10**].

5.12. Calibrated submanifolds

Let (X^{n+r}, g) be a Riemannian manifold and let $M^n \subset X$ be a submanifold (possibly with boundary). Recall that M is *minimal* if any $x \in M$ is contained in an open neighborhood $U \subset M$ such that if V is an open subset with $\overline{V} \subset U$ and V' is a submanifold of X with $\partial V = \partial V'$, then $\operatorname{vol} V' \geq \operatorname{vol} V$. M is minimal iff its mean curvature vector is identically zero. M is said to be *minimizing* if for all $x \in M$, we may take $U = M$.

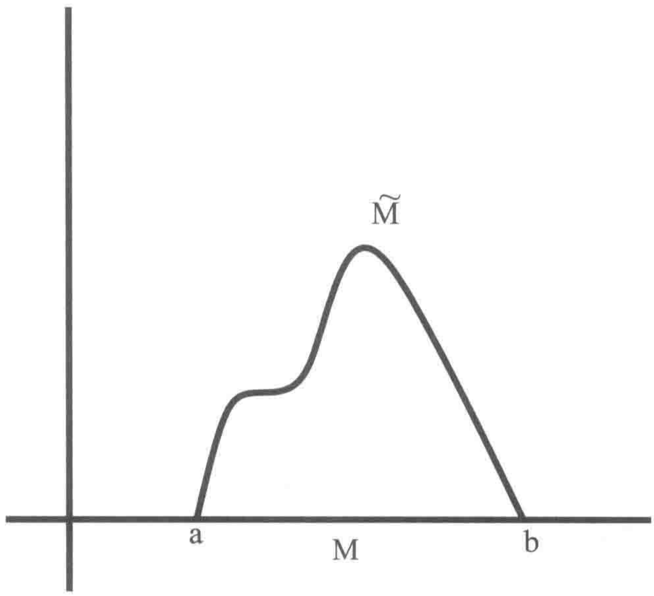
How can one construct minimizing submanifolds or even prove that a given submanifold is minimizing? There is no known general method. In this section we will describe a method for studying special classes of minimizing submanifolds. We begin with the simplest case:

A line is the shortest path between two points. Let $p, q \in \mathbb{E}^2$ be two points. How does one prove that the shortest path between p and q is the line segment connecting them?

Let $M(t)$ denote the parametrized line segment from a to b and let $\widetilde{M}(t)$ denote any rectifiable path from a to b. Without loss of generality, use coordinates (x^1, x^2) such that the line segment is along the x^1-axis.

We have

$$dx^1(M'(t)) = |M'(t)|,$$
$$dx^1(\tilde{M}'(t)) \leq |\tilde{M}'(t)|.$$



Letting $\phi = dx^1$, we have

$$\text{vol}(M) = \int_M \phi = \int_{\tilde{M}} \phi \leq \text{vol}(\tilde{M}), \tag{5.28}$$

where the middle equality follows from Stokes' Theorem.

The form ϕ, which enables us to show that M is minimizing, is an example of a calibration.

Calibrations. Throughout this section we use $\mathbf{G}(n, TX)$ to denote the Grassmannian of *unit volume* oriented n-planes in the tangent spaces of a Riemannian manifold X. That is, we only represent an n-plane by vectors $v_1 \wedge \ldots \wedge v_n$ such that $|v_1 \wedge \ldots \wedge v_n| = 1$, where the norm at each point is induced from the Riemannian metric. We will also use $\mathbf{G}(n, m) \subset \Lambda^n \mathbb{R}^m$ to denote unit volume n-planes in $\mathbb{R}^m$.

Definition 5.12.1 (Harvey and Lawson [**75**])**.** Let (X^{n+r}, g) be a Riemannian manifold. A *calibration* $\phi \in \Omega^n(X)$ is a differentiable n-form such that

i. $d\phi = 0$, and

ii. $\phi(E) \leq 1$ for all $E \in \mathbf{G}(n, TX)$.

A submanifold $i : M \to X$ is *calibrated* by ϕ if $i^*(\phi) = \text{dvol}_M$, where dvol_M is the volume form induced by the immersion.

Theorem 5.12.2 (Fundamental lemma of calibrations)**.** *Calibrated submanifolds are minimizing.*

Proof. Use equation (5.28). □

Remark 5.12.3. The condition to be calibrated corresponds to satisfying a first-order system of PDE, while being minimal (mean curvature vector identically zero) corresponds to satisfying a second-order system of PDE.

Given a calibration ϕ, what does it calibrate? The answer can be determined by setting up and solving an exterior differential system. We will set up two different types of systems for calibrated submanifolds of constant-coefficient calibrations. In what follows we will use linear Pfaffian systems, while in Chapter 7 we will use non-Pfaffian systems, once we have the general form of the Cartan-Kähler Theorem at our disposal.

Let $\phi \in \Omega^n(X)$ be a calibration, and let $x \in X$. We define

$$\mathrm{Face}_x(\phi) := \{E \in \mathbf{G}(n, T_xX) \mid \phi(E) = 1\}.$$

Let $\mathrm{Face}(\phi) \subset \mathbf{G}(n, TX)$ be the corresponding bundle. $M^n \subset X$ is calibrated by ϕ iff $\gamma(M) \subset \mathrm{Face}(\phi)$, where $\gamma : M \to \mathbf{G}(n, TX)$ is the Gauss map $x \mapsto (x, T_xM)$.

Remark 5.12.4. Let $\phi \in \Omega^n(\mathbb{E}^{n+r})$ have constant coefficients in a linear coordinate system. Then, after an appropriate normalization, ϕ is a calibration. In this situation, we may think of $\mathrm{Face}(\phi)$ as contained in $\mathbf{G}(n, n+r)$ (since $\mathrm{Face}_x(\phi)$ is independent of x), and the EDS is considerably easier to set up.

Complex submanifolds of a Kähler manifold are minimizing. Let (X, g, J, ω) be a Kähler manifold, that is, a complex manifold with a Riemannian metric g, symplectic form ω and the compatibilities $g(v, Jw) = \omega(v, w)$ and $g(Jv, Jw) = g(v, w)$ for all tangent vectors at all points.

For example, in $\mathbb{C}^m = \mathbb{E}^{2m}$ with standard coordinates (x^j, y^j), $1 \le j \le m$, if we let $z^j = x^j + \sqrt{-1}y^j$ (so that $J(dx^j) = -dy^j, J(dy^j) = dx^j$), then

$$\omega = \frac{i}{2}(dz^1 \wedge d\overline{z^1} + \ldots + dz^m \wedge d\overline{z^m}).$$

Proposition 5.12.5. *Let (X, ω) be Kähler and let $\phi = \frac{1}{k!}\omega^{\wedge k}$. Then ϕ is a calibration and $\mathrm{Face}_x(\phi) = G(\mathbb{C}^k, \mathbb{C}^m) \subset G(2k, T_xX)$. In other words, $\mathrm{Face}_x(\phi) = \{E \in G(2k, T_xX) \mid J(E) = E\}$.*

Wirtinger observed that if v, w is an oriented orthonormal basis of a plane E, then $\omega(v, w) \le 1$ with equality iff E is a complex line. Federer generalized this to ϕ, and Harvey and Lawson's general definition was inspired by the observation that Wirtinger's inequality implies that complex submanifolds are minimizing.

Proof of Wirtinger's inequality, case $k = 1$. Let E have orthonormal basis v, w. Recall the Cauchy-Schwarz inequality that $\langle x, y\rangle \le |x||y|$, with

equality iff $y = \lambda x$ for some $\lambda \in \mathbb{R}^+$:

$$\omega(v, w) = \langle v, J(w)\rangle \leq |v||Jw| = |v||w|.$$

One has equality in the Cauchy-Schwarz inequality iff $J(w) = v$, which implies E is complex. For the general case, see [**74**]. $\square$

We will now discuss four calibrations discovered by Harvey and Lawson [**75**], the *special Lagrangian calibration*, the *associative calibration*, the *coassociative calibration* and the *Cayley calibration*. Each produces a large class of minimizing submanifolds. These manifolds have recently become important for the study of mirror symmetry and M-theory [**41**]. Associative and coassociative calibrations exist for manifolds with G_2 holonomy, and the Cayley calibration exists for manifolds with $Spin_7$ holonomy (see Chapter 8 for a discussion of holonomy).

Special Lagrangian submanifolds. Let $\mathbb{R}^{2m} = \mathbb{C}^m$ have its standard complex structure J and flat metric $\langle\, ,\rangle$. As explained in §A.3, we obtain a symplectic form ω on $\mathbb{R}^{2m}$ from this data. Then $E \in G(m, \mathbb{R}^{2m})$ is Lagrangian, i.e., $\omega|_E{=}0$, if and only if $J(E) \perp E$ (see [**74**]).

Using standard complex coordinates on $\mathbb{R}^{2m} = \mathbb{C}^m$, we define a complex-valued m-form $dz = dz^1 \wedge \ldots \wedge dz^m$.

Proposition 5.12.6. *Given $E \in \mathbf{G}(m, 2m)$ (the unit volume Grassmannian) we have $|dz(E)| \leq 1$, with equality if and only if E is Lagrangian.*

Proof. We will show that

$$|dz(E)| = |E \wedge J(E)|.$$

The result will then follow by Hadamard's inequality (A.5).

Write $\mathbb{R}^{2m} = V$ and let $e_1, \ldots, e_m, Je_1, \ldots, Je_m$ be an orthonormal basis of V, where $e_j = \frac{\partial}{\partial x^j}$, and let $\epsilon_1, \ldots, \epsilon_m$ be an orthonormal basis of E. There exists $A \in \mathrm{End}(V)$ such that $A(e_i) = \epsilon_i$ and $A(Je_i) = J\epsilon_i$. In fact, such an A is complex-linear because it commutes with J. Let $E_0 = e_1 \wedge \ldots \wedge e_m$. We have $\rho(A)(E_0) = E$, where $\rho(A) : \Lambda^m V \to \Lambda^m V$ is the induced linear mapping. Therefore, $|dz(E)| = |dz(\rho(A)E_0)| = |\det_{\mathbb{C}}(A) dz(E_0)| = |\det_{\mathbb{C}}(A)|$. $\square$

Since dz is complex-valued, it cannot be a calibration, but we can form calibrations from it: Let

$$\alpha = \mathsf{Re}(dz) = \mathsf{Re}(dz^1 \wedge \ldots \wedge dz^m) \in \Omega^m(\mathbb{C}^m).$$

Corollary 5.12.7. *α is a calibration, called the special Lagrangian calibration.*

The proof of Proposition 5.12.6 shows that in fact $\alpha_\theta = \mathsf{Re}(e^{i\theta} dz)$ is a calibration for all $\theta \in [0, 2\pi)$. From this proof, we also deduce $\mathrm{Face}(\alpha)$:

Corollary 5.12.8. Face(α) *is the* $SU(m)$*-orbit of* E_0.

We now derive a linear Pfaffian system for special Lagrangian manifolds.

By Appendix A, we may write, with respect to the basis e_j, Je_j above,

$$\mathfrak{su}(m) = \left\{ \begin{pmatrix} \kappa & -\psi \\ \psi & \kappa \end{pmatrix} \mid \kappa \in \mathfrak{so}(m), {}^t\psi = \psi, \mathrm{trace}(\psi) = 0 \right\}.$$

We set up an EDS on $ASU(m) = \mathcal{F}_{SU(m)} \to \mathbb{R}^{2m}$ which has Maurer-Cartan form

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_{n+k} \\ \omega^{n+l} & \omega^{n+l}_j & \omega^{n+l}_{n+k} \end{pmatrix},$$

where $\omega^i_j = -\omega^j_i = \omega^{n+i}_{n+j}$, $\omega^{n+k}_l = \omega^{n+l}_k = -\omega^k_{n+l}$, and $\sum_i \omega^{n+i}_i = 0$.

Let $I = \{\omega^{n+i}\}$, $J = \{\omega^i, \omega^{n+i}\}$. The tableau determined by the (ω^{n+l}_j) has characters $s_1 = n, s_2 = n-2, \ldots, s_{n-1} = 2, s_n = 0$.

Exercise 5.12.9: Show that (I, J) is involutive.

Associative submanifolds. Let $\mathbb{R}^7$ be equipped with a nondegenerate positive 3-form $\phi \in \Lambda^3\mathbb{R}^7$, or more generally let M^7 be a Riemannian manifold with holonomy G_2, and call the induced 3-form ϕ as well. (Such forms are discussed in §A.5, and holonomy is discussed in Chapter 8.)

The form ϕ is a calibration, called the *associative calibration*, and its calibrated manifolds are called associative manifolds. The name comes from the fact that ϕ induces a structure of the imaginary octonions on $\mathbb{R}^7$ and the three-planes calibrated by ϕ are exactly those isomorphic to a copy of the imaginary quaternions, i.e., where the octonionic multiplication is associative (see Example 7.5.7 for details).

There exists a basis $e_1, \ldots, e_7$ for $\mathbb{R}^7$ such that

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} \rho(B) & -{}^tA \\ A & B \end{pmatrix} \middle|\, B \in \mathfrak{so}(4), \rho(B) \in \mathfrak{so}(3) \right\}, \tag{5.29}$$

where the first two columns of A are arbitrary but the third column is dependent on the first two (see Exercises A.5.3). As in the special Lagrangian case, ϕ-calibrated manifolds are the integral manifolds of a linear Pfaffian system given by $I = (\omega^\mu)$ and $J = (\omega^\mu, \omega^j)$, where $1 \le j \le 3$ and $4 \le \mu \le 7$. The tableau of the system has the form A, and we have already seen that such a tableau must be involutive (set $n = 3$ in Exercise 4.5.9.1). We conclude:

Proposition 5.12.10. *The Pfaffian system for associative manifolds is involutive with character* 3 *and Cartan integer* 4.

Coassociative submanifolds. We may also consider the calibration $\psi := *\phi$ on $\mathbb{E}^7$.

Exercise 5.12.11: Set up and perform Cartan's Test for coassociative submanifolds.

Cayley submanifolds. One can similarly define a calibration 4-form on $\mathbb{R}^8$ related to the $Spin_7$ action on $\mathbb{R}^8$. The form is called the *Cayley form.* One arrives at a similar EDS for Cayley manifolds as follows: Here

$$\mathfrak{spin}_7 = \begin{pmatrix} \mathfrak{sp}(1)_1 + \rho(\mathfrak{sp}(1)_2) & -{}^tA \\ A & \mathfrak{sp}(1)_3 + \tilde{\rho}(\mathfrak{sp}(1)_2) \end{pmatrix},$$

where the columns of A satisfy $J_1\pi_1 + J_2\pi_2 + J_3\pi_3 - \pi_4 = 0$ and $\rho, \tilde{\rho}$ are both isomorphic to the standard representations, but given in terms of different, nonstandard bases.

Setting up the system as for associative manifolds, the tableau is

$$A = (u, v, w, \epsilon_1 u + \epsilon_2 v + \epsilon_3 w),$$

which is clearly involutive.

Chapter 6

Applications to PDE

Introduction. Consider the well-known closed-form solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ due to d'Alembert:

$$u(x,t) = f(x+ct) + g(x-ct), \tag{6.1}$$

where f and g are arbitrary C^2 functions. It is rare that all solutions of a given PDE are obtained by a single formula (especially, one which does not involve integration). The key to obtaining the d'Alembert solution is to rewrite the equation in "characteristic coordinates" $\eta = x + ct, \xi = x - ct$, yielding $u_{\eta\xi} = 0$. By integrating in η or in ξ, we get

$$u_\eta = F(\eta), \qquad u_\xi = G(\xi),$$

where F and G are independent arbitrary functions; then (6.1) follows by another integration. For which other PDE do such solution formulas exist? And, how can we find them in a systematic way?

In this chapter we study invariants of exterior differential systems that aid in constructing integral manifolds, and we apply these to the study of first and second-order partial differential equations, and classical surface theory. (For second-order PDE, we will restrict attention to equations for one function of two variables.) All functions and forms are assumed to be smooth.

In §6.1 we define *symmetry vector fields* and in particular *Cauchy characteristic vector fields* for EDS. We discuss the general properties of Cauchy characteristics, and use them to recover the classical result that any first-order PDE can be solved using ODE methods. In §6.2 we define the *Monge characteristic systems* associated to second-order PDE, and discuss *hyperbolic exterior differential systems.* In §6.3 we discuss a systematic

method, called *Darboux's method*, which helps uncover solution formulas like d'Alembert's (when they exist), and more generally determines when a given PDE is solvable by ODE methods. We also define the *derived systems* associated to a Pfaffian system.

In §6.4 we treat Monge-Ampère systems, focusing on several geometric examples. We show how solutions of the sine-Gordon equation enable us to explicitly parametrize surfaces in $\mathbb{E}^3$ with constant negative Gauss curvature. We show how consideration of complex characteristics, for equations for minimal surfaces and for surfaces of constant mean curvature (CMC), leads to solutions for these equations in terms of holomorphic data. In particular, we derive the Weierstrass representation for minimal surfaces and show that any CMC surface has a one-parameter family of non-congruent deformations.

In §6.5 we discuss integrable extensions and Bäcklund transformations. Examples discussed include the Cole-Hopf and Miura transformations, the KdV equation, and Bäcklund's original transformation for pseudospherical surfaces. We also prove Theorem 6.5.14, relating Bäcklund transformations to Darboux-integrability.

6.1. Symmetries and Cauchy characteristics

Infinitesimal symmetries. One of Lie's contributions to the theory of ordinary differential equations was to put the various solution methods for special kinds of equations in a uniform context, based on *infinitesimal symmetries* of the equation—i.e., vector fields whose flows take solutions to solutions. This generalizes to EDS when we let the vector fields act on differential forms via the Lie derivative operator $\mathcal{L}$ (see Appendix B):

Definition 6.1.1. Let $\mathcal{I}$ be an EDS on Σ. A vector field $\mathsf{v} \in \Gamma(T\Sigma)$ is an (infinitesimal) symmetry of $\mathcal{I}$ if $\mathcal{L}_{\mathsf{v}}\psi \in \mathcal{I}$ for all $\psi \in \mathcal{I}$.

Exercises 6.1.2:

1. The Lie bracket of any two symmetries is a symmetry; thus, symmetries of a given EDS form a Lie algebra. ◎
2. To show that v is a symmetry, it suffices to check the condition in Definition 6.1.1 on a set of forms which generate $\mathcal{I}$ differentially. ◎
3. Let $\mathcal{I}$ be the Pfaffian system generated by the contact form $\theta = dy - z\,dx$ on $\mathbb{R}^3$. Verify that

$$\mathsf{v} = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + (g_x + zg_y - zf_x - z^2 f_y)\frac{\partial}{\partial z}, \tag{6.2}$$

for any functions $f(x, y), g(x, y)$, is a symmetry. Then, find the most general symmetry vector field for $\mathcal{I}$. ◎

Given an integral manifold N_0 of an exterior differential system $\mathcal{I}$, a symmetry vector field v can be used to obtain a one-parameter family of integral manifolds, as follows. Let ϕ_t denote the one-parameter group of diffeomorphisms of Σ generated by v, and let

$$N_t := \phi_t(N_0). \tag{6.3}$$

Exercise 6.1.3: Show that if N_t is smooth, then it is an integral manifold of $\mathcal{I}$. ⊚

Remark 6.1.4. Cartan [**26, 35**] determined those exterior differential systems whose symmetries depend on arbitrary functions (like the contact symmetries in Exercise 6.1.2). Determining the symmetries usually amounts to solving an overdetermined system of differential equations, so most of the time the Lie algebra of symmetries is empty. An interesting exception is the Pfaffian system $\{dy - p\,dx,\ dp - q\,dx,\ dz - q^2 dx\}_{\mathsf{diff}}$ defined on $\mathbb{R}^5$ with coordinates x, y, z, p, q. In this case the symmetries form a 14-dimensional Lie algebra isomorphic to the split form of the exceptional simple Lie algebra $\mathfrak{g}_2$ (see Appendix A). Cartan [**27**] proved that this is the largest possible Lie algebra of symmetries for a rank three Pfaffian system with generic derived flag on a five-dimensional manifold.

Unfortunately, enumerating the symmetry vector fields for a given system of differential equations is usually harder than solving the system itself. For this reason, we refrain from further discussion of symmetries in general,[1] and instead focus on a class of symmetries which are easily calculated and allow us to reduce the dimension of the manifold on which the system is defined:

Cauchy characteristics.

Definition 6.1.5. Let $\mathcal{I}$ be an EDS on Σ. A vector field $\mathsf{v} \in \Gamma(T\Sigma)$ is a *Cauchy characteristic vector field* of $\mathcal{I}$ if $\mathsf{v} \lrcorner \psi \in \mathcal{I}$ for all $\psi \in \mathcal{I}$.

The flow lines of v are called *Cauchy characteristic curves* (or, simply, *Cauchy characteristics*) of $\mathcal{I}$.

Exercises 6.1.6:

1. If v is a Cauchy characteristic vector field for $\mathcal{I}$, then it is also a symmetry. ⊚
2. The Lie bracket of any two Cauchy characteristic vector fields is a Cauchy characteristic vector field. ⊚
3. Show that v is a Cauchy characteristic vector field if and only if $\mathsf{v} \lrcorner \psi \in \mathcal{I}$ for a set of forms ψ that generate $\mathcal{I}$ algebraically. ⊚

[1]For more on symmetries, see the monograph [**6**] or the textbook [**127**].

While calculating the symmetries of a given EDS involves solving differential equations for the components of the vector field, the condition in Definition 6.1.5 is algebraic (and linear) in the components of v with respect to a local basis. If v is a Cauchy characteristic vector field, so is any multiple of v by a smooth function. So, it would be more correct to speak of *Cauchy characteristic line fields*; however, for calculations it is easier to fix a vector field.

Example 6.1.7. On $\mathbb{R}^4$ with coordinates x, y, z, w, let $\mathcal{I} = \{\theta\}_{\mathsf{diff}}$ for $\theta = dy - z\,dx$. Suppose $\mathsf{v} = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} + c\dfrac{\partial}{\partial z} + e\dfrac{\partial}{\partial w}$. Because $d\theta = dx \wedge dz$, the condition $\mathsf{v} \lrcorner\, d\theta \equiv 0 \mod \theta$ implies that $a = c = 0$, while $\mathsf{v} \lrcorner\, \theta = 0$ implies that $b = 0$. Hence any Cauchy characteristic vector fields for this system are multiples of $\partial/\partial w$.

While the flow generated by a symmetry v preserves integral manifolds, when v is a Cauchy characteristic vector field more is true:

Proposition 6.1.8. *Let N_0 be an n-dimensional integral manifold for an EDS, and let v be a Cauchy characteristic vector field which is transverse to N_0. Let N_t be as in (6.3), and let S be the $(n+1)$-dimensional union of the integral manifolds N_t. Then S, when smooth, is an integral manifold of $\mathcal{I}$.*

Proof (for $\dim N_0 = 1$**).** Suppose N_0 is an integral curve of $\mathcal{I}$ with v transverse to N_0. In order for the surface S to be smooth, we must restrict t to the open interval of values for which v is transverse to N_t. Let $\Omega \in \mathcal{I}^2$ and let w be tangent to one of the N_t. Then $\Omega(\mathsf{v}, \mathsf{w}) = (\mathsf{v} \lrcorner\, \Omega)(\mathsf{w}) = 0$, because $\mathsf{v} \lrcorner\, \Omega \in \mathcal{I}$. □

Exercise 6.1.9: Generalize this argument to the case where N_0 has arbitrary dimension.

Example 6.1.10 (A first-order PDE). Let (x^1, x^2, p_1, p_2, u) be the usual coordinates on $J^1(\mathbb{R}^2, \mathbb{R})$ (see §1.9). The equation $u_t = u_x$ is equivalent to the EDS $\mathcal{I}$ obtained by restricting the contact system to the smooth hypersurface $\Sigma \subset J^1(\mathbb{R}^2, \mathbb{R})$ defined by $p_2 = p_1$. We will use $x = x^1, t = x^2, u$ and $p = p_1$ as coordinates on Σ, where the system is generated algebraically by $\theta := du - p(dx + dt)$ and $d\theta = -dp \wedge (dx + dt)$.

Suppose

$$\mathsf{v} := a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + c\frac{\partial}{\partial p} + e\frac{\partial}{\partial u}$$

is a Cauchy characteristic vector field on Σ. Then $\mathsf{v} \lrcorner\, \theta = 0$ implies that $e = (a+b)p$. Then $\mathsf{v} \lrcorner\, d\theta \equiv 0$ modulo θ implies that $c = 0$ and $a + b = 0$. Hence $\mathsf{v} = \dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial t}$, up to multiple.

Thus, integral surfaces of $\mathcal{I}$ are constructed by translating integral curves along the curves where u, p and $x+t$ are constant.

Example 6.1.11 (Surfaces in $\mathbb{E}^3$). Recall from §2.1 that the integral manifolds of $\mathcal{I} = \{\omega^3\}_{\mathsf{diff}}$ on $\mathcal{F}_{\mathsf{on}}(\mathbb{E}^3)$ are the first-order adapted lifts of surfaces $\Sigma^2 \subset \mathbb{E}^3$. This system admits a Cauchy characteristic vector field as follows:

Let $(\omega^j)^*, (\omega^i_j)^*$ be a basis of vector fields on $\mathcal{F}$ dual to the entries of the Maurer-Cartan form. Write $\mathsf{v} = a_j(\omega^j)^* + a^j_i(\omega^i_j)^*$. Then $\mathsf{v} \lrcorner\, \omega^3 = 0$ implies that $a_3 = 0$, and

$$\begin{aligned} \mathsf{v} \lrcorner\, d\omega^3 &= \mathsf{v} \lrcorner\, (-\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2) \\ &= a_1\omega^3_1 - a^1_3\omega^1 + a_2\omega^3_2 - a^2_3\omega^2 \end{aligned}$$

implies that $\mathsf{v} = (\omega^2_1)^*$ up to multiple. As explained in §1.8, v corresponds to infinitesimal rotations of the frame within the $\{e_1, e_2\}$-plane.

Solving First-Order PDE. Any first-order PDE for one unknown function can, in theory, be solved through ODE techniques, because (as we will show) the corresponding EDS always has a Cauchy characteristic vector field.

Suppose $u, x^1, \ldots, x^n, p_1, \ldots, p_n$ are the usual coordinates on $J^1(\mathbb{R}^n, \mathbb{R})$, and the PDE takes the form

$$F(u, x^1, \ldots, x^n, p_1, \ldots, p_n) = 0. \tag{6.4}$$

Let $\Sigma \subset J^1$ be the subset on which (6.4) holds. To ensure that Σ is a smooth manifold, and the equation is genuinely of first order, we will assume that $\partial F/\partial p_i \neq 0$ for some i at each point of Σ.

Let θ be the pullback to Σ of the contact form $du - p_1dx^1 - \ldots - p_ndx^n$. As explained in §1.9, solutions of the PDE (6.4) are in 1-to-1 correspondence with n-dimensional integral manifolds $i : N \to \Sigma$ of the Pfaffian system generated by θ, with the independence condition $i^*(dx^1 \wedge \ldots \wedge dx^n) \neq 0$.

Because $\dim \Sigma = 2n$, $\theta \wedge (d\theta)^n = 0$. Differentiating (6.4) gives

$$\begin{aligned} 0 = dF &= F_{x^i}dx^i + F_{p_j}dp_j + F_u du \\ &\equiv (F_{x^i} + F_u p_i)dx^i + F_{p_j}dp_j \mod \theta. \end{aligned} \tag{6.5}$$

Since this is the only relation between the differentials of the jet coordinates on Σ, $\theta \wedge (d\theta)^{n-1} \neq 0$ at each point. So, by the Pfaff Theorem 1.9.17, there are local coordinates on Σ^{2n} such that θ is a nonzero multiple of

$$dy^1 + y^2dy^3 + \ldots + y^{2n-2}dy^{2n-1}.$$

Thus, $\mathsf{v} = \partial/\partial y^{2n}$ gives a Cauchy characteristic. (Without reference to local coordinates, v can be described as the unique vector, up to multiple, such that $\mathsf{v} \lrcorner\, (\theta \wedge (d\theta)^{n-1} = 0)$.)

The Cauchy characteristic vector field enables us to reduce our PDE to an ODE. We do this by choosing an $(n-1)$-dimensional integral manifold L of $\mathcal{I}$ (corresponding to initial data), and then using flow by v to get an n-dimensional integral manifold. If F may be solved for p_n, then v is transverse to hyperplanes where x^n is constant. Thus, we obtain a starting manifold L by letting $x^n = C$, $u = f(x^1, \ldots, x^{n-1})$ for an arbitrary function, and $p_i = \partial f/\partial x^i$. To obtain a solution to the original PDE we only need to integrate the ODE system corresponding to flow by v.

Example 6.1.12 (The inviscid Burger's equation $u_t = uu_x$)**.** Using the same coordinates on $J^1(\mathbb{R}^2, \mathbb{R})$ as in Example 6.1.10, our equation becomes $p_2 - up_1 = 0$, defining a smooth four-dimensional manifold $\Sigma \subset J^1$. On Σ, use x, t, u and $p = p_1$ as local coordinates; then $\theta = du - p\,dx - up\,dt$ and

$$d\theta \equiv -(dp - p^2 dt) \wedge (dx + udt) \mod \theta. \tag{6.6}$$

Any Cauchy characteristic vector field v must be annihilated by θ and by both factors on the right-hand side of (6.6). Thus, we may take

$$\mathsf{v} = \frac{\partial}{\partial t} - u\frac{\partial}{\partial x} + p^2\frac{\partial}{\partial p}.$$

The flow generated by v is obtained by solving the ODE system

$$dx/dt = -u, \qquad du/dt = 0, \qquad dp/dt = p^2.$$

This system has solution $x(t) = x_0 - ut$, $u(t) = u_0$, $p(t) = 1/(p_0^{-1} - t)$.

To construct an integral surface, we may start with an integral curve of θ at $t = 0$ by setting $x_0 = s$, $u_0 = f(s)$ and $p_0 = f'(s)$, where f is an arbitrary C^1-function. By flowing along the Cauchy characteristics we obtain the implicit solution

$$u = f(s) = f(x + ut). \tag{6.7}$$

Note that the implicit function theorem for (6.7) fails when $tf'(s) = 0$. When this happens, the flow under v has caused p to become undefined, and the graph of u as a function of x develops a vertical tangent. At later times, the surface defined by (6.7) may not be a graph over the xt-plane.

Remark 6.1.13. In standard PDE texts, what we have discussed is called the *method of characteristics.* In some treatments (e.g., [**85**]) the characteristic curves in $J^1(\mathbb{R}^2, \mathbb{R})$ are described as curves in $J^0(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^3$ with a tangent plane attached at each point, and these are known as *characteristic strips.* The solution surface in $\mathbb{R}^3$ is then formed by the envelope of those characteristic strips that pass through the initial data.

Exercises 6.1.14:

1. Consider the PDE $u_x^2 + u_y^2 = 2u$.

(a) Show that, when x, y, p, q are used as coordinates, the Cauchy characteristic line field is generated by $\mathsf{v} = p(\partial_p + \partial_x) + q(\partial_q + \partial_y)$.
(b) Find all regular solutions satisfying the initial condition $u(x, 0) = \frac{2}{5}x^2$. ◎

2. Solve $\frac{1}{2}(u_x^2 + u_y^2) + xu_x + yu_y = u$ subject to $u(x, 0) = (1 - x^2)/2$. ◎
3. ([**85**], Ch. 1) Solve the PDE $u_y = u_x^3$ with the initial condition $u(x, 0) = 2x^{3/2}$. Show that the only solutions which are regular over all of $\mathbb{R}^2$ are linear functions of x and y. ◎

Remark 6.1.15. A general first-order *system* of PDE does not admit a Cauchy characteristic line field. For example, suppose a system of two equations for two functions $u(x, y), v(x, y)$ defines a smooth submanifold $\Sigma^6 \subset J^1(\mathbb{R}^2, \mathbb{R}^2)$ that submerses onto $J^0(\mathbb{R}^2, \mathbb{R}^2)$. The contact system restricts to be a system $\mathcal{I}$ on Σ generated algebraically by a pair of 1-forms θ_1, θ_2 and a pair of linearly independent 2-forms. As we will see shortly, if there were Cauchy characteristics, this would imply that $\mathcal{I}$ is locally a pullback to Σ of a system defined on a lower-dimensional quotient manifold—in this case, at most dimension five. This would in turn require that the 2-forms are decomposable, modulo the θ's, and share a common factor. It is easy to see that, generically, this is not the case.

Exercise 6.1.16: Show that the only quasilinear systems for u, v which have nontrivial Cauchy characteristics are those that uncouple into two first-order equations with the same characteristics.

Retracting spaces. Because the condition in Definition 6.1.5 is C^∞-linear in v, the Cauchy characteristics form a well-defined distribution $\mathcal{A}(\mathcal{I}) \subset T\Sigma$ given by

$$\mathcal{A}(\mathcal{I})_x := \{\mathsf{v} \in T_x\Sigma | \mathsf{v} \lrcorner \mathcal{I} \subset \mathcal{I}\}, \qquad x \in \Sigma.$$

It follows from Exercise 6.1.6 that $\mathcal{A}(\mathcal{I})$ is closed under Lie brackets. The dimension of $\mathcal{A}(\mathcal{I})$ is upper semicontinuous on Σ (i.e., in an open neighborhood of a point, it can only decrease), and so we may (and will) restrict our attention to an open set in Σ on which it is constant. The dual of $\mathcal{A}(\mathcal{I})$ is a Pfaffian system known as the *retracting space* or *Cartan system* of $\mathcal{I}$:

$$\mathcal{C}(\mathcal{I})_x := \{\theta \in T_x^*\Sigma | \mathsf{v} \lrcorner \theta = 0, \forall \mathsf{v} \in \mathcal{A}(\mathcal{I})_x\}.$$

If θ is a section of $\mathcal{C}(\mathcal{I})$ and $\mathsf{v}_1, \mathsf{v}_2 \in \mathcal{A}(\mathcal{I})$, then using the "vector field definition" of the exterior derivative (see Appendix B) we get $d\theta(\mathsf{v}_1, \mathsf{v}_2) = 0$, and this implies that $\mathcal{C}(\mathcal{I})$ is Frobenius. The Frobenius Theorem implies that there exist local coordinates $x^1, \ldots, x^n$ such that $\mathcal{C}(\mathcal{I}) = \{dx^1, \ldots, dx^s\}$. Then $\mathcal{I}$ can be defined using only the variables $x^1, \ldots, x^s$.

Proposition 6.1.17. *The retracting space $\mathcal{C}(\mathcal{I})$ is the smallest Frobenius system such that $\Lambda\,\mathcal{C}(\mathcal{I})$ contains a set of algebraic generators for $\mathcal{I}$.*

Proof. Let J be a Frobenius system such that there exists a finite collection of forms $\psi_1, \ldots, \psi_m \in \Lambda J$ which generate $\mathcal{I}$ algebraically. In terms of these generators, any form $\Phi \in \mathcal{I}$ can be written as

$$\Phi = \sum_{k=1}^{m} \beta_k \wedge \psi_k.$$

Let v be any vector annihilated by J. Then $\mathsf{v} \lrcorner \Phi = \sum_{k=1}^{m} (\mathsf{v} \lrcorner \beta_k) \wedge \psi_k$, which belongs in $\mathcal{I}$. Hence $\mathsf{v} \in \mathcal{A}(\mathcal{I})$. Since v and Φ are arbitrary, we conclude that $\mathcal{C}(\mathcal{I}) \subset J$.

Furthermore, the smallest such J cannot be larger than $\mathcal{C}(\mathcal{I})$. For, suppose $\mathcal{C}(\mathcal{I})$ has codimension $l > 0$ within J, and let $\pi_1, \ldots, \pi_l$ be linearly independent forms in J with $\pi_k \notin \mathcal{C}(\mathcal{I})$. Partition the generators for $\mathcal{I}$ into $\varphi_1, \ldots, \varphi_m \in \Lambda\mathcal{C}(\mathcal{I})$, plus generators $\psi_1, \ldots, \psi_n$ involving the π_k. If v is any vector in $\mathcal{A}(\mathcal{I})$, and ψ_1 has lowest degree among the ψ's, then $\mathsf{v} \lrcorner \psi_1$ must be expressible in terms of the φ's. In particular, any term in ψ_1 involving one of the π_k may be omitted since that term already is generated by the φ's. It follows that we may take all the generators to be in $\Lambda\mathcal{C}(\mathcal{I})$. □

The above proposition implies that we may calculate the retracting space simply by obtaining a minimal set of algebraic generators for the ideal.

Exercises 6.1.18:

1. Consider an EDS generated by a single 1-form $\theta \in \Omega^1(\Sigma)$. Show that the dimension of $\mathcal{C}(\mathcal{I})$ will be $2k+1$, where k is the largest integer such that $\theta \wedge (d\theta)^k|_\Sigma \neq 0$. ◎

2. ([**63**], §126) Consider the system of two second-order PDE given by $u_{xx} + \frac{1}{3}u_{xy}^3 = 0$ and $u_{xy}u_{yy} = 1$. Let $\mathcal{I}$ be the pullback of the standard contact system to the six-dimensional submanifold $\Sigma \subset J^2(\mathbb{R}^2, \mathbb{R})$ defined by these equations. Show that $\mathcal{C}(\mathcal{I})$ is five-dimensional, and give a Cauchy characteristic vector field for $\mathcal{I}$. ◎

Quotient spaces. Assume $\mathcal{I}$ is an EDS defined on Σ, and $\mathcal{C}(\mathcal{I})$ has constant rank k. Then the maximal integral manifolds of $\mathcal{C}(\mathcal{I})$ form a foliation of Σ by smooth submanifolds of codimension k. In some cases, the leaf space of this foliation is a manifold, and is referred to as the *quotient by the Cauchy characteristics.*

Example (6.1.11 continued). For our tautological EDS for lifts of surfaces, $\mathcal{C}(\mathcal{I})$ is the Frobenius system on $\mathcal{F}$ spanned by $\omega^1, \omega^2, \omega^3, \omega^3_1, \omega^3_2$. Integral curves of $\mathcal{C}(\mathcal{I})$ form a foliation of $\mathcal{F}$. The leaf space for this foliation is actually a five-dimensional homogeneous space which we identify with $U = UT\mathbb{E}^3$, the unit tangent bundle of $\mathbb{E}^3$, by letting x be the basepoint and e_3 the unit vector in $T_x\mathbb{R}^3$.

Whenever a smooth quotient by the Cauchy characteristics exists, the system $\mathcal{I}$ is generated algebraically by the pullback of a well-defined system on the quotient manifold. This fact is a consequence of the following more general result:

Proposition 6.1.19. *Let $\pi : \Sigma \to Q$ be a smooth fibration. Let $I \subset \Lambda^p T^*\Sigma$ be a subbundle of rank k spanned by p-forms that are semi-basic for π. Suppose that for any $\psi \in \Gamma(I)$ and any vector field v on Σ that is tangent to the fibers of π,*

$$\mathcal{L}_{\mathsf{v}}\psi \in \Gamma(I). \tag{6.8}$$

Then there is a well-defined subbundle $\widetilde{I} \subset \Lambda^p T^*Q$ such that a p-form $\widetilde{\psi}$ is a section of $\widetilde{I}$ if and only if $\pi^*\widetilde{\psi} \in \Gamma(I)$.

Let $\mathfrak{J}$ be the algebra of wedge products of forms that are semi-basic for π. (Note that $\mathfrak{J}$ is not an exterior ideal.) Then condition (6.8) is equivalent to

$$d\psi \equiv 0 \mod \mathfrak{J}^{p+1} + I \wedge T^*\Sigma, \tag{6.9}$$

i.e., $d\psi$ is a sum of wedge products of semi-basic forms and wedge products with sections of I.

Proof of 6.1.19. Let $x \in \Sigma$, and let U be an open set containing $\pi(x)$ such that $\sigma : U \to \Sigma$ is a local section of the fibration and $N = \pi(U)$ passes through x. Let $\widetilde{I} \subset \Lambda^p T^*\Sigma|_U$ be the image of $I|_N$ under σ^*.

To show that $\widetilde{I}$ is independent of choices, let v be tangent to the fibers and let $\phi_t : \Sigma \to \Sigma$ be the 1-parameter family of diffeomorphisms generated by v. (If necessary, we restrict the domain of ϕ_t to some open set around x, containing N, so that ϕ_t is defined for all t in an open interval about $t = 0$.) Then (6.8) implies that $\phi_t^*\psi \in \Gamma(I)$, and thus $(\phi_t \circ \sigma)^*\psi \in \Gamma(\widetilde{I})$ for any t. □

Theorem 6.1.20. *If $\mathcal{I}$ is an EDS on Σ and the leaf space of the Cauchy characteristics is a smooth manifold Q, then there is an EDS $\widetilde{\mathcal{I}}$ on Q such that $\mathcal{I} = \{\pi^*\widetilde{\mathcal{I}}\}_{\mathsf{alg}}$, where π is the projection onto the leaf space.*

Proof. Let $J = \mathcal{C}(\mathcal{I})$, and let I^p_J be the bundle spanned by semi-basic p-forms in $\mathcal{I}$. Then Proposition 6.1.19 implies that I^p_J is the pullback of the degree p piece of a differential ideal $\widetilde{\mathcal{I}}$ on Q (see the exercises below).

Because $\Gamma(I^p_J) \subset \mathcal{I}^p$, we have $\{\pi^*\widetilde{\mathcal{I}}\}_{\mathsf{alg}} \subset \mathcal{I}$. Moreover, since $\mathcal{I}$ has a set of algebraic generators that are semi-basic (see Proposition 6.1.17), $\mathcal{I} \subset \{\pi^*\widetilde{\mathcal{I}}\}_{\mathsf{alg}}$. □

In practice, one exploits the existence of the quotient manifold and its EDS to obtain a better geometric understanding of the system. However, it may be easier to continue to calculate using the coordinates or coframing on the original space. For example, if the original space is a submanifold of a jet bundle, or a Lie group, then these spaces come equipped with coframes for which the structure equations are particularly simple.

Exercises 6.1.21:

1. Show that $\widetilde{\mathcal{I}}$, as constructed degree by degree in the proof of Theorem 6.1.20, is actually a differential ideal on Q.

2. Let Σ, π, Q and $\mathfrak{J}$ be as above. Suppose Q has dimension $2m$, and Ω is a 2-form in $\mathfrak{J}$ such that $\Omega^m \neq 0$. Under what conditions is Ω the pullback of a well-defined symplectic form on Q? ◎

6.2. Second-order PDE and Monge characteristics

In this section we define the characteristic systems associated to hyperbolic second-order PDE. Characteristic systems are connected to the characteristic variety defined in Chapter 4, and will play a key role in our discussion of Darboux's method. (They should not be confused with Cauchy characteristics, defined earlier.) We will also generalize these systems to a class of EDS known as *hyperbolic exterior differential systems.*

In Example 5.5.3, we calculated the tableau (5.15) for the rank 3 Pfaffian system I obtained by restricting the standard contact forms

$$\begin{aligned}
\theta_0 &= dz - p\,dx - q\,dy,\\
\theta_1 &= dp - r\,dx - s\,dy,\\
\theta_2 &= dq - s\,dx - t\,dy
\end{aligned}$$

to the hypersurface $\Sigma \subset J^2(\mathbb{R}^2, \mathbb{R})$ defined by a regular second-order PDE

$$F(x, y, z, p, q, r, s, t) = 0. \tag{6.10}$$

Applying Cartan's Test to the tableau shows that such systems are involutive with solutions depending locally on $s_1 = 2$ functions of one variable. Although there is usually no explicit way to describe the solutions in terms of these functions, we will see several examples where such a description exists when we discuss Darboux's method.

We saw that a nonzero covector $\xi = \xi_1 dx + \xi_2 dy$ belongs to the characteristic variety of I if and only if

$$F_r\xi_1^2 + F_s\xi_1\xi_2 + F_t\xi_2^2 = 0. \tag{6.11}$$

Thus, the characteristic directions are the null lines for this quadratic form.

Definition 6.2.1. The PDE (6.10) is said to be *elliptic*, *hyperbolic* or *parabolic* according to whether the determinant $F_r F_t - \frac{1}{4}F_s^2$ of the quadratic form is positive, negative or zero, respectively (cf. [**85**]).

For the rest of this discussion we will assume the determinant is nonzero at every point of Σ. In fact, we will assume the PDE is hyperbolic; however, everything we do may be carried out in the elliptic case, by using complex-valued forms (see §6.4).

Hyperbolic PDE and characteristic systems. Note that (6.11) implies that a linear combination of the system 2-forms is congruent, modulo I, to a decomposable 2-form with ξ as one of the factors. Therefore, when the PDE is hyperbolic, there are two linearly independent decomposable 2-forms in the ideal that are independent of the θ's. We may choose linearly independent forms $\pi_1, \pi_2, \omega_1, \omega_2$ that are not in $\mathcal{I}^1$, such that $\pi_1 \wedge \omega_1$ and $\pi_2 \wedge \omega_2$ are these decomposable forms in $\mathcal{I}^2$, and $\omega_1 \wedge \omega_2 \neq 0$ is the independence condition. In order to simplify the tableau, we may also choose new forms $\tilde{\theta}_1, \tilde{\theta}_2$ in $\mathcal{I}^1$ so that

$$d\tilde{\theta}_1 \equiv -\pi_1 \wedge \omega_1, \quad d\tilde{\theta}_2 \equiv -\pi_2 \wedge \omega_2 \mod I. \tag{6.12}$$

(However, this is not immediately necessary.)

At each point of an integral surface for I, the tangent vectors to the surface that are annihilated by ω_1 or ω_2 are characteristic in the sense defined above. Moreover, since each π_i pulls back to the surface to be a multiple of ω_i, each such vector is annihilated by all of the forms of one of the following two *characteristic systems*:

$$\mathcal{M}_1 := \{\theta_0, \theta_1, \theta_2, \pi_1, \omega_1\}, \quad \mathcal{M}_2 := \{\theta_0, \theta_1, \theta_2, \pi_2, \omega_2\}.$$

Each integral surface is foliated by integral curves of $\mathcal{M}_1$ and by integral curves of $\mathcal{M}_2$. In order to distinguish them from Cauchy characteristics, these curves are sometimes called *Monge characteristics* [**143**].

Remark 6.2.2. For parabolic equations, there is only one decomposable 2-form and only one characteristic system. In classical terminology, one says that in this case the Monge characteristics are *confounded.*

Remark 6.2.3. The characteristic systems are diffeomorphism invariants of the exterior differential system $(\Sigma, \mathcal{I})$. For, suppose two Pfaffian systems $(M, \mathcal{I})$ and $(N, \mathfrak{J})$, both arising from regular second-order PDE for one function of two variables, are contact-equivalent, which means that there is a diffeomorphism $\phi : M \to N$ such that $\phi^* \mathfrak{J} = \mathcal{I}$, and such ϕ respects the independence conditions. Then ϕ takes integral planes to integral planes, and tableau to tableau.

Exercises 6.2.4:
1. Assume that $F_r \neq 0$. Using (6.11), show that $\xi = dy - m\,dx$ belongs to the characteristic variety if and only if m is a root of the quadratic equation

$$F_r m^2 - F_s m + F_t = 0. \tag{6.13}$$

Assuming that this equation has distinct roots m_1, m_2, show that the decomposable 2-forms in $\mathcal{I}$ are

$$(dy - m_1 dx) \wedge (\pi_2 + m_2\pi_3), \qquad (dy - m_2 dx) \wedge (\pi_2 + m_1\pi_3),$$

where π_1, π_2, π_3 are as in the structure equations (5.13), (5.14).
2. For the hyperbolic PDE $s = pq$ (i.e., $z_{xy} = z_x z_y$), show that the characteristic systems are

$$\mathcal{M}_1 = \{\theta_0, \theta_1, \theta_2, dr - q(r + p^2)dy, dx\},$$
$$\mathcal{M}_2 = \{\theta_0, \theta_1, \theta_2, dt - p(t + q^2)dx, dy\}.$$

Remark 6.2.5. Note that if the PDE (5.12) is quasilinear, then the equation (6.13) defining the characteristics only involves the variables x, y, z, p, q. More generally, we say that a second-order PDE for one function of two variables has *first-order characteristics* if there exists a rank 3 Pfaffian system on $J^1(\mathbb{R}^2, \mathbb{R})$ whose pullback to Σ is contained in one of the characteristic systems of $\mathcal{I}$. This is the case when (5.12) is a Monge-Ampère equation (see §6.4); for other examples, see [**63**], Ch. IV. The existence of first-order characteristics is intrinsic to the system, since $J^1(\mathbb{R}^2, \mathbb{R})$ is the quotient of Σ by the foliation dual to the retracting space of I's first derived system (see §6.3).

Hyperbolic Exterior Differential Systems. We now generalize the notion of Monge characteristics to systems which have structure equations similar to (6.12).

Definition 6.2.6 ([**22**]). A *hyperbolic EDS of class k* is an EDS algebraically generated by k independent 1-forms and two decomposable 2-forms with no common factors.

Thus, the retracting space of such a system has dimension $k + 4$. For this reason, a hyperbolic EDS of class k may be assumed to be defined on a $(k + 4)$-dimensional manifold.

This generalization will be useful for studying hyperbolic PDE because the prolongation of a hyperbolic system of class k is hyperbolic of class $k+2$. We show this in the special case of systems arising from second-order hyperbolic PDE, which have class $k = 3$:

Example 6.2.7. Let $\mathcal{I}$ be the rank 3 Pfaffian system on $\Sigma^7 \subset J^2(\mathbb{R}^2, \mathbb{R})$ corresponding to a second-order hyperbolic equation, as defined in the first

part of this section. Any integral element satisfying the independence condition is annihilated by

$$\theta_3 := \pi_1 - h_1\omega_1, \qquad \theta_4 := \pi_2 - h_2\omega_2$$

for some parameters h_1, h_2. Then these h_i can be taken as coordinates along fibers of the bundle $\Sigma' = \mathcal{V}_2(\mathcal{I}, \Omega)$ of integral planes. The prolongation of $\mathcal{I}$ is the Pfaffian system on Σ' generated by $I' = \{\theta_0, \dots, \theta_4\}$. (Again, we omit pullbacks from the notation.)

The structure equations of the prolongation are

$$d\theta_0 \equiv 0, \quad d\theta_1 \equiv 0, \quad d\theta_2 \equiv 0, \quad d\theta_3 \equiv -\pi_3 \wedge \omega_1, \quad d\theta_4 \equiv -\pi_4 \wedge \omega_2 \mod I',$$

where $\pi_3 := dh_1 + (A_1 - h_1 B_1)\omega_2$ and $\pi_4 := dh_2 + (A_2 - h_2 B_2)\omega_1$, and A_j, B_j are functions on Σ' such that $d\pi_j \equiv A_j \omega_1 \wedge \omega_2$ and $d\omega_j \equiv B_j \omega_1 \wedge \omega_2$ modulo I'.

Notice that these structure equations have the same form as the structure equations (6.12) for $\mathcal{I}$. Therefore, $\mathcal{I}'$ is a hyperbolic EDS of class five. We define the characteristic systems of the prolongation as $\mathcal{M}_1' = \{\theta_0, \dots, \theta_4, \pi_3, \omega_1\}$ and $\mathcal{M}_2' = \{\theta_0, \dots, \theta_4, \pi_4, \omega_2\}$. Each characteristic system of $\mathcal{I}$ pulls back to lie inside one of the characteristic systems of the prolongation (see Example 6.3.16 below).

In general, for a hyperbolic EDS of class k we define two characteristic systems of rank $k+2$ in the same manner as in the example, by adjoining to the 1-forms of the system the factors of one of the two decomposables.

Hyperbolic systems of even class arise when we consider systems of equations for two functions of two variables:

Exercise 6.2.8: Show that a system of the form $u_x = f(v), v_y = g(u)$, when converted to an EDS on a codimension-two submanifold of $J^1(\mathbb{R}^2, \mathbb{R}^2)$, gives a hyperbolic EDS of class two. Show that it may also be encoded by a hyperbolic EDS of class zero (i.e., generated by a pair of decomposable 2-forms) defined on $J^0(\mathbb{R}^2, \mathbb{R}^2)$. ◎

6.3. Derived systems and the method of Darboux

We now turn to *Darboux's method*, which gives a systematic way of discovering how to solve a second-order PDE using ODE techniques. If the equation passes a certain test, then we may specify two arbitrary functions of one variable, and then construct solutions by solving systems of ODE. (In fact, for some equations Darboux's method can even lead us to a formula giving all solutions in terms of two arbitrary functions.) The test is carried out by calculating the *derived flag* for each of the characteristic systems

defined in the previous section. Accordingly, we first need to discuss such constructions:

Derived Systems. One argument for using exterior differential systems to study PDE is that it guides one to a good framing that highlights the geometric aspects of the system in question. For a Pfaffian system, computing the derived systems shows us one way to adapt the coframe to the geometry of the system.

Let $I \subset T^*\Sigma$ be a Pfaffian system of constant rank. The *derived system* $I^{(1)}$ of I is the subsystem spanned by forms in I whose exterior derivatives are zero modulo I. More explicitly, let $\delta : \Gamma(I) \to \Gamma(\Lambda^2(T^*\Sigma/I))$ be defined by $\delta(\theta) = d\theta \mod I$. Because δ is $C^\infty(\Sigma)$-linear, it is induced by a vector bundle homomorphism from I to $\Lambda^2(T^*\Sigma/I)$. Since the coefficients of δ, when expressed in matrix form, are smooth, its rank is upper semi-continuous, so we may restrict to an open set in Σ on which it has constant rank. On this open set, we define $I^{(1)} := \ker \delta$.

If $I^{(1)} = I$ then I is Frobenius (cf. Definition 1.3.5); otherwise, $I^{(1)}$ is smaller than I. In that case, we define a sequence of systems $I^{(k+1)} := (I^{(k)})^{(1)}$, until $I^{(N)}$ either is a Frobenius system or has rank zero. The *derived flag* of I is defined as

$$I = I^{(0)} \underset{\neq}{\supset} I^{(1)} \underset{\neq}{\supset} \dots \underset{\neq}{\supset} I^{(N)}$$

and N is called the *derived length* of I. The dimensions of the systems in the derived flag are differential invariants and can be used to classify Pfaffian systems into types.

Remark 6.3.1. These dimensions must satisfy certain inequalities. For example, if $r_k = \dim I^{(k)} - \dim I^{(k+1)}$ and $m = \dim \mathcal{C}(I) - \dim I$, then

$$\begin{aligned}
r_0 &\leq \tfrac{1}{2}m(m-1),\\
r_1 &\leq mr_0 + \tfrac{1}{2}r_0(r_0-1),\\
r_2 &\leq r_1(r_0+m) + \tfrac{1}{2}r_1(r_1-1),\\
&\dots
\end{aligned}$$

In fact, for generic systems these inequalities are equalities. See [**53**] for further inequalities and invariants of Pfaffian systems.

Example 6.3.2. On $\mathbb{R}^4$, with coordinates u, v, x, y, let I be spanned by the 1-forms

$$\theta_1 := dy - vdx, \qquad \theta_2 := dv + xdu.$$

The above inequalities imply that the first derived system is at least one-dimensional. In fact, since $d\theta_1 = dx \wedge dv \equiv xdu \wedge dx$ modulo I and $d\theta_2 =$

$dx \wedge du$, then $I^{(1)}$ is spanned by

$$\omega := \theta_1 + x\theta_2 = dy + xdv - vdx + x^2 du.$$

Because $d\omega = 2dx \wedge \theta_2$, $I^{(1)}$ isn't Frobenius. Hence the derived flag terminates in $I^{(2)} = 0$.

The generic Pfaffian system I of rank two on a four-dimensional manifold Σ has derived length two. Such systems are said to define an *Engel structure* on Σ. Like contact structures and symplectic structures, all Engel structures are locally diffeomorphic (cf. the Engel Normal Form Theorem in [**20**]).

Exercise 6.3.3: On $\mathbb{R}^5$, with coordinates u, v, x, y, z, let I be spanned by the 1-forms

$$\theta_1 := dx - xdu, \qquad \theta_2 := dz + udx + vdy, \qquad \theta_3 := dy - udv.$$

Show that $I^{(1)} = \{\theta_1, \theta_2\}$ and that $I^{(2)}$ is Frobenius. ◎

Remark 6.3.4. By contrast, a generic Pfaffian system of rank three in five variables has derived length two but contains no Frobenius system. Unlike Engel structures, such systems are not necessarily locally diffeomorphic to each other. Their classification is the subject of Cartan's famous "five variables paper" [**27**].

Exercise 6.3.5: Suppose $\mathcal{I}$ is a hyperbolic EDS of class k, $\mathcal{I}'$ is its prolongation, and I' is the Pfaffian system spanned by the 1-forms of $\mathcal{I}'$. Show that $(\mathcal{I}')^{(1)}$ is spanned by the pullbacks of the 1-forms of $\mathcal{I}$.

Darboux's Method. Let $\mathcal{I}$ be the exterior differential system constructed in §6.2, encoding a hyperbolic second-order PDE for one function of two variables. On any integral surface of $\mathcal{I}$ satisfying the independence condition, the Monge characteristic curves form two foliations which are transverse at each point of the surface. Suppose one characteristic system $\mathcal{M}_i$ happens to contain two linearly independent 1-forms that are exact derivatives:

$$dW, dX \in \mathcal{M}_i.$$

Then W and X are both constant along the integral curves of $\mathcal{M}_i$. (Any such function, whose differential lies in one of the $\mathcal{M}_i$, is known as a *Riemann invariant* for the PDE.) It follows that W and X are functionally related on the integral surface. If we suppose that $\{dW, dX\} \subset \mathcal{M}_i$ is linearly independent from I, we can assume that, say, dX restricts to be nonzero on an open set in the integral surface. Then W must be some function of X on the surface. When both characteristic systems have these properties, the functions may be arbitrarily specified, and used to construct the solution.

Example 6.3.6. For Liouville's equation $z_{xy} = e^z$, the Pfaffian system I is defined by

$$\begin{aligned} \theta_0 &= dz - p\,dx - q\,dy \\ \theta_1 &= dp - r\,dx - e^z dy \\ \theta_2 &= dq - e^z dx - t\,dy \end{aligned} \quad \text{with} \quad \left.\begin{aligned} d\theta_1 &\equiv -(dr - pe^z dy) \wedge dx \\ d\theta_2 &\equiv -(dt - qe^z dx) \wedge dy \end{aligned}\right\} \bmod I.$$

The characteristic system $\mathcal{M}_1$ contains two exact derivatives, dx and $d(r - \frac{1}{2}p^2)$, the latter obtained by adding multiples of dx and θ_1 to $dr - pe^z dy$. Similarly, dy and $d(t - \frac{1}{2}q^2)$ are in $\mathcal{M}_2$. Hence,

$$r - \tfrac{1}{2}p^2 = f(x), \qquad t - \tfrac{1}{2}q^2 = g(y) \tag{6.14}$$

for some functions that depend on the particular solution of Liouville's equation defining our integral surface.

Again, we may use these functions to determine the solution. For, if we choose f and g arbitrarily, and restrict I to the codimension-two submanifold defined by (6.14), then I becomes a Frobenius system. This means that we can solve for $z(x, y)$ by solving systems of ODE. In this case, we can solve the system

$$\partial z/\partial x = p, \qquad \partial p/\partial x = f(x) + \tfrac{1}{2}p^2 \tag{6.15}$$

and a similar system in the y-direction, and be guaranteed by the Frobenius condition that we have generated a solution to the original PDE.

Definition 6.3.7. A hyperbolic exterior differential system $\mathcal{I}$ is said to be *Darboux-integrable* if each characteristic system $\mathcal{M}$ contains a Frobenius system Δ_i of rank two that is independent from I.

Proposition 6.3.8. *Let $\mathcal{I}$ be a Darboux-integrable system defined on Σ. Let W_1, W_2, X_1, X_2 be functions defined on an open set $U \subset \Sigma$ such that $W_i, X_i \in \Delta_i$ for $i = 1, 2$. Let f_1, f_2 be arbitrary smooth functions and let $N \subset U$ be the codimension-two submanifold defined by $W_i = f_i(X_i)$. Then $\mathcal{I}|_N$ is Frobenius.*

Proof. Since $\mathcal{M}_i = \mathcal{I}^1 \oplus \Delta_i$, then on U the image $\delta(I)$ is spanned by $dW_1 \wedge dX_1$ and $dW_2 \wedge dX_2$. □

Example (6.3.6 continued)**.** We have seen that Liouville's equation is Darboux-integrable. Now, to obtain a solution, it seems that we must solve Riccati differential equations (6.15) for p and q. While this is easy to do numerically, there is no general formula for representing solutions of a Riccati equation by quadratures. However, since $f(x)$ is arbitrary, we may make convenient choices for the form of $f(x)$. In particular, if we set $f(x) = F'(x) - \frac{1}{2}F(x)^2$, then $F(x)$ is another solution of the Riccati equation that

p satisfies. It is standard (cf. [**140**]) that $v = 1/(p - F(x))$ satisfies a linear differential equation. Solving that equation using an integrating factor leads to

$$p = \frac{X''}{X'} - \frac{2X'}{X+Y},$$

where X and Y are arbitrary functions of x and y respectively, and $F(x) = X''/X'$. Differentiating gives

$$z = \ln \frac{2X'Y'}{(X+Y)^2}. \tag{6.16}$$

Proposition 6.3.9. *The general solution to Liouville's equation $z_{xy} = e^z$ is* (6.16), *where $X(x)$ and $Y(y)$ are arbitrary functions.*

Exercise 6.3.10: Fill in the details in the derivation of solution formula (6.16).

The two extra equations (6.14) which we impose on the solution of Liouville's equation are compatible with it, in the sense that the restriction of I to the submanifold defined by them is involutive—in fact, it is Frobenius. (In classical terminology, any extra equation that defines a submanifold to which the system restricts to be involutive is called an "integral" for the given PDE. If the extra equation is of lower order than the given PDE then it is an "intermediate integral", and otherwise it is a "general integral".) The Frobenius condition means that the solution is uniquely determined, up to a choice of constants, by the two arbitrary functions in (6.14). Thus, we have a concrete realization of what Cartan's Test predicts for this system, namely that the solution depends on two functions of one variable.

Exercise 6.3.11: Prove that every solution of Liouville's equation arises from integrating a Frobenius system determined by the equation and (6.14).

Example 6.3.12. For the PDE $z_{xx} + \frac{1}{2}z_{xy}^2 = 0$, the characteristic equation (6.13) takes the form $m^2 - s\,m = 0$. The characteristic systems are

$$\mathcal{M}_1 = \{\theta_0, \theta_1, \theta_2, dy, ds + sdt\}, \quad \mathcal{M}_2 = \{\theta_0, \theta_1, \theta_2, dy - s\,dx, ds\}.$$

It is clear that $\mathcal{M}_1$ contains the rank two Frobenius system $\{dy, (ds/s) + dt\}$; so, like in (6.14), we let

$$t + \ln s = \varphi(y)$$

for an arbitrary function φ. Meanwhile, the derived flag of $\mathcal{M}_2$ is

$$\begin{aligned}\mathcal{M}_2' &= \{\theta_0, \theta_1, dy - s\,dx, ds\},\\ \mathcal{M}_2'' &= \{dy - s\,dx, dp - \tfrac{1}{2}s^2 dx, ds\},\end{aligned}$$

terminating in a Frobenius system of rank three. Using s as a characteristic coordinate, we let

$$y - sx = \psi(s)$$

for an arbitrary function ψ. We may use these equations to determine x and t in terms of s and y. Then, since $\theta_0, \theta_1, \theta_2$ span a Frobenius system, we can determine p, q and z by integration. For example,

$$dp = r dx + s dy = d(rx + sy) - x\, dr - y\, ds = d(rx + sy) - \psi(s) ds$$

gives

$$p = rx + sy - \Psi(s) = \frac{s}{2}(y + \Psi'(s)) - \Psi(s),$$

where Ψ is an antiderivative of ψ.

Exercise 6.3.13: Determine q and z in terms of y and s in this example. ◎

Remark 6.3.14. Just as the above PDE (abbreviated in classical notation as $r + \frac{1}{2}s^2 = 0$) is Darboux-integrable, so is any equation of the form $r + f(s) = 0$ ([**63**], §154).

Exercises 6.3.15:
1. Show that $r - qs + pt = 0$ is Darboux-integrable.
2. Determine for which functions $f(v)$ and $g(u)$ the system of Exercise 6.2.8 is Darboux-integrable. ◎
3. Let $m_1 \neq m_2$ be the roots of the characteristic equation (6.13) for a PDE of the form $r + f(s,t) = 0$.
(a) Show that each characteristic system contains one of the 1-forms $ds + m_i dt$, and that both of these forms are integrable.
(b) Show that the equation is Darboux-integrable if m_1 and m_2 both satisfy the inviscid Burger's equation $\partial m/\partial t = m\, \partial m/\partial s$. Express this condition as an overdetermined system of PDE for $f(s,t)$, and investigate the corresponding EDS.

Note that Definition 6.3.7 applies to hyperbolic systems of arbitrary rank. For example, a second-order PDE for $z(x,y)$ may become Darboux-integrable after prolonging k times. Then, as before, the Pfaffian system becomes a Frobenius system of rank $2k+3$ when restricted to submanifolds of the form $U = f(X)$, $V = g(Y)$, where $\{dU, dX\}$ and $\{dV, dY\}$ are the rank two systems described in the definition.

Example 6.3.16 (Darboux integrability after one prolongation)**.** Consider the PDE $s = pz$ (i.e., $z_{xy} = z_x z$). The characteristic systems are

$$\mathcal{M}_1 = \{\theta_0, \theta_1, \theta_2, dx, dr - (p^2 + rz)dy\},$$
$$\mathcal{M}_2 = \{\theta_0, \theta_1, \theta_2, dy, dt - p(z^2 + q)dx\}.$$

Since $\theta_2 - z\theta_0 = d(q - \frac{1}{2}z^2)$ modulo dy, $\mathcal{M}_2$ contains a rank two Frobenius system. However, the derived flag of $\mathcal{M}_1$ terminates in dx; to continue checking for Darboux integrability, we must prolong.

Because the mixed partials of z can be expressed in terms of lower derivatives, for higher prolongations we need only add variables $p_k = \partial^k z/\partial x^k$ and $q_k = \partial^k z/\partial y^k$ for $k > 2$. For the first prolongation we adjoin 1-forms

$$\theta_3 = dr - (p^2 + rz)dy - p_3 dx,$$
$$\theta_4 = dt - p(z^2 + q)dx - q_3 dy.$$

Since the dy-characteristic system contains the pullback of $\mathcal{M}_2$ from above, we already know that it contains a rank 2 Frobenius system. Meanwhile, the dx-characteristic system is

$$\mathcal{M}_1 = \{\theta_0, \dots, \theta_4, dx, dp_3 - (3rp + zp_3)dy\}.$$

The derived flag of $\mathcal{M}_1$ ends, after five steps, with the Frobenius system

$$\mathcal{M}_1^{(5)} = \left\{dx, d\left(\frac{p_3}{p} - \frac{3}{2}\left(\frac{r}{p}\right)^2\right)\right\}.$$

Hence we may impose the equations

$$q - \tfrac{1}{2}z^2 = \varphi(y), \qquad \frac{p_3}{p} - \frac{3}{2}\left(\frac{r}{p}\right)^2 = \psi(x) \tag{6.17}$$

for arbitrary functions φ and ψ. Then $\theta_0, \theta_1, \theta_2, \theta_3$ restrict to give a Frobenius system on the submanifold defined by these equations. (The full prolongation system $\theta_0, \dots, \theta_4$ becomes integrable if we add the condition $t - zq = \varphi'(y)$, obtained by differentiating (6.17).)

Exercises 6.3.17:

1. Show that $s = pq/(x-y)^n$ is Darboux-integrable when $n = 0$, and Darboux-integrable after one prolongation when $n = 1$.
2. Show that $s + n(n-1)z/(x-y)^2$, where n is a positive integer, is Darboux-integrable only after n prolongations ([**42**] IV, Ch.3).

The problem of classifying which second-order PDE become Darboux-integrable after a finite number of prolongations has a long history and is still a subject of current research [**5, 88**].

One of the earliest results in this area is Lie's proof that the only equations of the form $s = f(u)$ that are Darboux-integrable after a finite number of prolongations are those with $f(u) = \exp(au + b)$; see [**63**], §170 for a proof. Another significant early result was Goursat's classification [**64**] of quasilinear equations $s = f(x, y, u, p, q)$ that are Darboux-integrable without prolongation. This classification was later extended, using symmetry methods, by E. Vessiot; see [**149**] for more details.

Recently, some progress has been made on the related problem of determining which systems of even class are Darboux-integrable [**22**]. As in Exercise 6.2.8, such systems can arise from a system of first-order PDE for two functions $u(x, y), v(x, y)$ and its prolongations.

Remark 6.3.18 (Semi-integrability). If a rank-two Frobenius system is present in only one of the two characteristic systems of a hyperbolic EDS, it is said to be *semi-integrable* by the method of Darboux. In this case, one can still construct solutions for the system using ODE techniques alone. For simplicity, we will explain how this is done for a single second-order hyperbolic PDE (cf. [**149**], Proposition 3.3):

For $dW, dX \in \mathcal{M}_1$, the imposed equation $W = f(X)$ defines a six-dimensional submanifold $N \subset \Sigma$ on which the pullback of $\mathcal{I}$ has a Cauchy characteristic vector field annihilated by $\mathcal{M}_2$. Thus, we can construct an integral surface of $\mathcal{I}$ by starting with an integral curve K transverse to these Cauchy characteristics, and taking the union of the characteristics through K. Constructing these characteristics involves solving a system of first-order ODE for four unknowns which, along with the transversality condition, depend on the choice of f.

Remark 6.3.19 (Darboux for parabolic PDE). Suppose that, for a parabolic second-order equation, the single characteristic system $\mathcal{M}$ contains a rank two Frobenius system. Then $\mathcal{M}$ itself is Frobenius (see [**143**], §16.1). The contact system contains a well-defined rank two subsystem J, linearly independent from θ_0, such that $\mathcal{M} = \mathcal{C}(J)$. By Theorem 6.1.20, J is well-defined on the quotient manifold Q by the leaves of $\mathcal{M}$. One can construct integral surfaces of I by starting with an integral curve $N \subset Q$ of J, and then writing θ_0 as a contact form on the inverse image of N, which is three-dimensional. Thus, integral surfaces may again be constructed by ODE methods.

6.4. Monge-Ampère systems and Weingarten surfaces

In this section we will study a class of second-order PDE's for which one can define an EDS on a smaller-dimensional manifold instead of the usual seven-dimensional manifold of §6.2. These equations, called *Monge-Ampère equations*, are of the form

$$A z_{xx} + 2B z_{xy} + C z_{yy} + D + E(z_{xx} z_{yy} - z_{xy}^2) = 0, \tag{6.18}$$

where A, B, C, D, E are functions of x, y, z, z_x, z_y. (As in §6.2, we assume that the partial derivatives of the left-hand side with respect to z_{xx}, z_{xy} and z_{yy} are never simultaneously zero.) Later in this section, we also will examine a geometrically natural class of surfaces, called *linear Weingarten surfaces*, which are locally equivalent to solutions of a Monge-Ampère differential equation.

Exercise 6.4.1: Show that the PDE (6.18) is elliptic or hyperbolic, in the sense of §6.2, if $AC - B^2 - DE$ is positive or negative, respectively.

Let $\theta = dz - p\,dx - q\,dy$, the contact form on $J^1(\mathbb{R}^2, \mathbb{R})$. On an integral surface of θ, satisfying the independence condition $dx \wedge dy \neq 0$, we have $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Then $z(x, y)$ satisfies (6.18) if and only if the surface is also an integral of the 2-form

$$(6.19)\quad \Psi := A\,dp \wedge dy + B(dq \wedge dy - dp \wedge dx) \\ - C\,dq \wedge dx + D\,dx \wedge dy + E\,dp \wedge dq.$$

Therefore, we may encode (6.18) by the EDS $\mathcal{I} = \{\theta, \Psi\}_{\mathsf{diff}}$. (Note that this is not the only possible choice of generators; since $d\theta = -(dp \wedge dx + dq \wedge dy)$, the B-term in (6.19) may be replaced by $2Bdq \wedge dy$ or by $-2Bdp \wedge dx$.) The prolongation of $\mathcal{I}$ gives the usual rank 3 Pfaffian system described in §6.2.

Remark 6.4.2. Similarly, a quasi-linear third-order evolution equation for one function of two variables may be encoded by an EDS on a manifold of lower dimension than $J^3(\mathbb{R}^2, \mathbb{R})$; see Example 6.5.6.

The following definition captures the essential features of our system $\mathcal{I}$ on $J^1(\mathbb{R}^2, \mathbb{R})$:

Definition 6.4.3. A *Monge-Ampère system* on a five-dimensional manifold Σ is an exterior differential system which is generated differentially by a 1-form θ of Pfaff rank five and a 2-form Ψ that is linearly independent from $d\theta$, modulo θ.

Exercises 6.4.4:

1. Show that $d\Psi \equiv 0$ modulo $\theta, d\theta$ and Ψ. ◎
2. Show that if (6.18) is hyperbolic in the sense of Definition 6.2.1, then the corresponding Monge-Ampère system $\mathcal{I}$ is a hyperbolic EDS of class one.

Example 6.4.5. Laplace's equation $z_{xx} + z_{yy} = 0$ is equivalent to the Monge-Ampère system on $J^1(\mathbb{R}^2, \mathbb{R})$ generated by θ and $\Psi = dp\wedge dy - dq\wedge dx$.

Example 6.4.6. The sine-Gordon equation

$$(6.20)\qquad u_{xy} = \sin u \cos u$$

is equivalent to the Monge-Ampère system on $J^1(\mathbb{R}^2, \mathbb{R})$ generated by θ and $\Psi = (dp - \sin u \cos u\, dy) \wedge dx$. (The usual version of the sine-Gordon equation, $u_{xy} = \sin u$, is equivalent to (6.20) by doubling u.) Note that $\Psi + d\theta$ is also decomposable.

It is no accident that these examples of Monge-Ampère systems are derived from Monge-Ampère equations. In fact, any Monge-Ampère system is locally equivalent to one generated by such an equation:

Proposition 6.4.7. *Let Σ^5 carry a Monge-Ampère system $\mathcal{I}$. In a neighborhood of any point in Σ, there are local coordinates x, y, z, p, q such that $\mathcal{I}$ is generated by the form $\theta' = dz - p\,dx - q\,dy$ and a 2-form of the form* (6.19) *for some functions A, B, C, D, E.*

Exercise 6.4.8: Prove this proposition. ◎

Example 6.4.9. Using the coordinate expression (1.3) for the mean curvature of the graph of $z(x, y)$, we see that the Monge-Ampère system corresponding to $H = 0$ has

$$\Psi = (1 + q^2)dp \wedge dy + pq(dp \wedge dx - dq \wedge dy) - (1 + p^2)dq \wedge dx.$$

Similarly, we can encode the PDE (1.4) for graphs with Gauss curvature $K = 1$ by a Monge-Ampère system with

$$\Psi = dp \wedge dq - (1 + p^2 + q^2)^2 dx \wedge dy.$$

These conditions on the curvatures H, K of a surface are examples of Weingarten equations, and are more naturally modeled by an EDS on the coframe bundle (see below).

The characteristic systems for a Monge-Ampère equation may be defined in a way similar to §6.2. Namely, suppose $\mathcal{I}$ is hyperbolic, i.e., we can find two decomposable 2-forms in $\mathcal{I}$ which are linearly independent modulo θ, and such that $\mathcal{I} = \{\theta, \omega_1 \wedge \pi_1, \omega_2 \wedge \pi_2\}_{\text{alg}}$. Then their respective factors form the two characteristic systems $\mathcal{M}_i = \{\theta, \omega_i, \pi_i\}$. As explained in §6.2, these characteristics systems pull back to lie in the characteristic systems of the prolongation.

Exercise 6.4.10: Find the characteristic systems for the Monge-Ampère system for the equation $z_{xx}z_{yy} - z_{xy}^2 = -1$. Show that, as a hyperbolic system of class one, it is Darboux-integrable . ◎

Remark 6.4.11. The method of Darboux, when applied to a Monge-Ampère system which is semi-integrable (see 6.3.18) is known as Monge's method ([**49**], Ch. XVI).

Exercise 6.4.12: Determine which Monge-Ampère equations of the form $z_{xy} = f(x, y, z, z_y)$ are integrable by Monge's method. ◎

Weingarten Surfaces.

Definition 6.4.13. A *Weingarten surface* is a surface in Euclidean space whose mean curvature and Gauss curvature satisfy some prescribed equation $F(H, K) = 0$. A *linear Weingarten equation* is a prescribed relationship $AK + 2BH + C = 0$, where A, B, C are constants.

Suppose $S \subset \mathbb{E}^3$ is a smooth surface satisfying a linear Weingarten equation, and $f : S \to \mathcal{F}$ is a first-order adapted framing along S. Then the framing gives an integral surface for the 1-form ω^3 on $\mathcal{F}$, and for the 2-form

$$\Psi := A\omega^3_1 \wedge \omega^3_2 + B(\omega^3_1 \wedge \omega^2 - \omega^3_2 \wedge \omega^1) + C\omega^1 \wedge \omega^2.$$

Exercise 6.4.14: Verify that $f^*\Psi = 0$, using the formulas (2.4) for the pullbacks of the ω^3_i.

The Weingarten equation is thus equivalent to an EDS $\mathcal{I}$ on $\mathcal{F}$ generated by Ψ, ω^3 and

$$\Theta := -d\omega^3 = \omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2.$$

(The linear Pfaffian system described in Example 5.8.2, which encodes the same linear Weingarten equation, is the prolongation of this EDS.)

As in Example 6.1.11, the retracting space for $\mathcal{I}$ is spanned by $\{\omega^1, \omega^2, \omega^3, \omega^1_3, \omega^2_3\}$, and the Cauchy characteristic curves are the fibers of the submersion from $\mathcal{F}$ to the five-dimensional manifold which is the unit tangent bundle $U = UT\mathbb{E}^3$:

$$\begin{tikzcd} \mathcal{F} \arrow[dr] \arrow[dd] & \\ & U \arrow[dl] \\ \mathbb{E}^3 & \end{tikzcd}$$

According to Theorem 6.1.20, there is a well-defined EDS on U which pulls back to $\mathcal{I}$ and is of Monge-Ampère type. (In fact, the structure equations of $\mathcal{F}$ imply that ω^3, $d\omega^3$ and all three terms of Ψ are pullbacks of well-defined forms on U.)

Exercise 6.4.15 (Parallel surfaces)**:** A point in U is determined by the basepoint $x \in \mathbb{E}^3$ and the unit vector e_3. Consider the map $F_r : U \to U$ defined by $(x, e_3) \mapsto (x + re_3, e_3)$.
(a) Show that F_r is a contact transformation, i.e., that $F_r^*\omega^3$ is a multiple of $\omega^3 = dx \cdot e_3$.
(b) Suppose that $M \subset U$ is an integral surface of ω^3 and of $\Psi = \omega^3_1 \wedge \omega^3_2 - r^2\omega^1 \wedge \omega^2$ and satisfies the independence condition $\omega^1 \wedge \omega^2 \neq 0$. Then M is a lift of a surface of constant Gauss curvature $K = 1/r^2$ in $\mathbb{E}^3$.

Show that $F_r(M)$ is a lift of a surface of constant mean curvature $H = 1/(2r)$ in $\mathbb{E}^3$. ◎

Thus, we recover the theorem of Bonnet, to the effect that each surface of constant positive Gauss curvature is parallel to two surfaces of constant mean curvature.

It follows that solutions of the PDE (1.4) for surfaces of constant positive Gauss curvature may be obtained by a contact transformation from solutions of the Monge-Ampère system for constant mean curvature (CMC) surfaces.

Moreover, since (as we will see below) any CMC surface admits a family of non-congruent isometric deformations, similar deformations are available for surfaces of constant positive Gauss curvature.

Exercises 6.4.16:

1. Show that $\mathcal{I}$ is elliptic or hyperbolic according to whether $B^2 - AC$ is positive or negative, respectively. ◎

2. Show that linear Weingarten equations for surfaces in a three-dimensional space form are also equivalent to Monge-Ampère systems. Which are hyperbolic? ◎

We finish this section with three extended examples that show how characteristics may be interpreted geometrically, and how to work with them when the system is elliptic.

Example 6.4.17 (Pseudospherical surfaces and the sine-Gordon equation). A *pseudospherical surface* is a surface $S \subset \mathbb{E}^3$ with Gauss curvature $K \equiv -1$. Setting $A = C = 1$ and $B = 0$ in the above EDS shows that, in this case, the 2-forms of $\mathcal{I}$ are spanned by the decomposable forms

$$\Psi + \Theta = (\omega^3_1 - \omega^2) \wedge (\omega^3_2 + \omega^1), \qquad \Psi - \Theta = (\omega^3_1 + \omega^2) \wedge (\omega^3_2 - \omega^1),$$

and the system is hyperbolic. Moreover, we may use the factors of the decomposable forms to express the second fundamental form as

$$\mathrm{II} = [\tfrac{1}{2}(\omega^3_1 + \omega^2) \circ (\omega^3_2 + \omega^1) - \tfrac{1}{2}(\omega^3_1 - \omega^2) \circ (\omega^3_2 - \omega^1)] \otimes e_3,$$

This shows that each characteristic curve projects to be an asymptotic line on S (see §2.8).

On an open subset of a pseudospherical surface that is free of umbilic points, we may choose a Darboux framing (see §2.3). This breaks the Cauchy characteristic symmetry of the EDS, but allows us to see how pseudospherical surfaces are connected to solutions of the sine-Gordon equation (6.20).

Because we are using Darboux frames,

$$\begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = \begin{pmatrix} \tan u & 0 \\ 0 & -\cot u \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where $u \in (0, \pi/2)$ is one-half of the angle between the asymptotic lines. We adjoin u as a new variable, and enlarge our EDS to a Pfaffian system $I = \{\omega^3, \omega^3_1 - (\tan u)\omega^1, \omega^3_2 + (\cot u)\omega^2\}$ on $\mathcal{F} \times \mathbb{R}$. Then a Darboux framing along a pseudospherical surface corresponds to an integral surface of $\mathcal{I} = \{I\}_{\mathsf{diff}}$ satisfying the independence condition $\omega^1 \wedge \omega^2 \neq 0$, and vice versa.

The vanishing of the exterior derivatives modulo I of the last two 1-forms of I implies that $\omega^1/\cos u$ and $\omega^2/\sin u$ pull back to be closed 1-forms on any integral surface. Hence there exist local coordinates t_1, t_2 such that $\omega^1 = \cos u\, dt_1$ and $\omega^2 = \sin u\, dt_2$.

Exercise 6.4.18: Verify that $\omega^1/\cos u$ and $\omega^2/\sin u$ are closed, and show that $x = (t_1 - t_2)/2$ and $y = (t_1 + t_2)/2$ are arclength coordinates along the asymptotic lines. This implies that the asymptotic lines form what is called a *Chebyshev net* [**43**] on the surface.

Substituting $du = u_x dx + u_y dy$ into the 2-forms of the EDS shows that $\omega_2^1 = u_x dx - u_y dy$ along any integral surface. Then it is easy to see that the structure equation $d\omega_2^1 = \omega_1^3 \wedge \omega_2^3$ implies that $u_{xy} = \sin u \cos u$.

Conversely, we may start with a solution to the sine-Gordon equation and produce a pseudospherical surface by integration. For, if we let $F = (e_1, e_2, e_3)$, then the structure equations $de_i = e_j\omega_i^j$ imply that

(6.21)
$$\frac{\partial F}{\partial x} = F \begin{pmatrix} 0 & u_x & -\sin u \\ -u_x & 0 & -\cos u \\ \sin u & \cos u & 0 \end{pmatrix} \qquad \frac{\partial F}{\partial y} = F \begin{pmatrix} 0 & -u_y & -\sin u \\ u_y & 0 & \cos u \\ \sin u & -\cos u & 0 \end{pmatrix}.$$

This is an overdetermined system for the matrix $F(x, y)$, and its integrability condition is the sine-Gordon equation for u. So, given a solution to sine-Gordon, we may obtain the framing by solving linear systems of ODE, and then solve for the surface $X(x, y) \in \mathbb{R}^3$ by integrating

$$\text{(6.22)} \qquad \frac{\partial X}{\partial x} = e_1 \cos u - e_2 \sin u, \qquad \frac{\partial X}{\partial y} = e_1 \cos u + e_2 \sin u.$$

For example, the traveling wave solution

$$\text{(6.23)} \qquad u = 2 \arctan\left(\exp(ax + a^{-1}y)\right), \qquad a \neq 0,$$

gives the standard pseudosphere when $a^2 = 1$ and gives Dini's surface when $a^2 \neq 1$ (see [**119**] for pictures).

Hilbert's Theorem states that pseudospherical surfaces in Euclidean space cannot be geodesically complete. Hence, even if the sine-Gordon solution is defined for all x and y, the corresponding surface will become singular. In fact, $\omega^1 \wedge \omega^2 = \sin 2u\, dx \wedge dy$ shows that the surface will be singular whenever u is a multiple of $\pi/2$, i.e., the asymptotic lines are tangent to each other.

Note that each of $x + y$ and $x - y$ is constant along one of the two sets of lines of curvature. This observation is useful if we suppose that S is a pseudospherical surface of revolution.

Exercise 6.4.19: Show that if S is a surface of revolution, the corresponding sine-Gordon solution can be assumed to be of the form $u = f(x + y)$. Then, determine all sine-Gordon solutions of this form. ◎

The classification of surfaces of revolution of constant Gauss curvature is due to Minding [**146**].

Example 6.4.20 (Minimal surfaces and the Weierstrass representation). Recall that a *minimal surface* is a surface $S \subset \mathbb{E}^3$ with mean curvature zero. Setting $A = C = 0$ and $B = 1$ in the EDS for a general linear Weingarten equation shows that the 2-forms of $\mathcal{I}$ are spanned by $\Theta = -d\omega^3$ and $\Psi = \omega^3_2 \wedge \omega^1 - \omega^3_1 \wedge \omega^2$. We will see below that studying the characteristic systems for $\mathcal{I}$ leads us to recover the classical *Weierstrass representation* (6.26) for minimal surfaces.

When we attempt to form the characteristic systems, we find that the decomposability equation

$$(\Theta + \lambda\Psi)^2 = 0$$

has roots $\lambda = \pm i$. So, we need to introduce complex-valued differential forms. For example, $\Theta - i\Psi = (\omega^3_1 - i\omega^3_2) \wedge (\omega^1 + i\omega^2)$, and similarly for the complex conjugate. If we define

$$\pi := \omega^3_1 - i\omega^3_2, \qquad \omega := \omega^1 + i\omega^2, \tag{6.24}$$

then the characteristic systems are $\mathcal{M} = \{\omega^3, \pi, \omega\}$ and $\overline{\mathcal{M}} = \{\omega^3, \overline{\pi}, \overline{\omega}\}$.

When we attempt to apply Darboux's method, we find that neither of these systems contains a rank 2 integrable subsystem. In fact, $d\pi \equiv 0$ modulo π, but $d\omega \equiv \overline{\pi} \wedge \omega^3$ modulo ω. However, this shows that $\mathsf{Re}\{\pi, \overline{\pi}\} = \{\omega^3_1, \omega^3_2\} =: J$ is a Frobenius system on $\mathcal{F}$. The quotient of $\mathcal{F}$ by the leaves of the distribution dual to J is a two-dimensional sphere, and the quotient map $\gamma : \mathcal{F} \to S^2$ is given by the unit vector e_3. Thus, the restriction of γ to an integral surface M of $\mathcal{I}$ is just the Gauss map of the minimal surface $\pi(M)$.

Because π drops to be well-defined (up to multiple) on S^2, we can define a complex structure on the sphere for which π spans the $(1,0)$-forms. Thus, we recover the well-known result that the Gauss map of a minimal surface S is holomorphic with respect to the complex structure on S defined by the $(1,0)$-form $\omega = \omega^1 + i\omega^2$.

It will be useful to fix a complex coordinate on the sphere. To do this, we will identify S^2 with the null quadric $N \subset \mathbb{CP}^2$ defined by $z_1^2 + z_2^2 + z_3^2 = 0$, by mapping e_3 to the vector $\mathsf{z} = e_1 - ie_2$. (This vector is associated to e_3 in a unique way, up to multiple, by the requirement that $e_3 \times \mathsf{z} = i\mathsf{z}$.) We then use stereographic projection

$$\mathsf{z} = [1 - w^2, i(1 + w^2), 2w], \qquad w \in \mathbb{C}, \tag{6.25}$$

to define a complex local coordinate w on N. Computing $d\mathsf{z}$ and comparing this with $d(e_1 - ie_2) = i(e_1 - ie_2)\omega^1_2 + e_3\pi$ shows that $\gamma^* dw$ is a multiple of π.

Exercise 6.4.21: Verify that, under our identification, the point (6.25) in N corresponds to the point in the unit sphere given by standard stereographic

projection

$$(x, y) \mapsto \left(2x, 2y, 1 - x^2 - y^2\right) / (1 + x^2 + y^2)$$

when $w = x + iy$. Thus, the complex structure on S^2 defined by π is the same as that given by the usual stereographic projection from $\mathbb{C}$.

Now suppose z is a local complex coordinate on our minimal surface S. Then $\omega = f(z)dz$ for a nonvanishing holomorphic function f, and $\gamma^* w = g(z)$ for some holomorphic function g. Notice that the point X on the surface satisfies

$$dX = e_1\omega^1 + e_2\omega^2 = \mathsf{Re}\left((e_1 - ie_2)(\omega^1 + i\omega^2)\right).$$

By substituting the expressions for $e_1 - ie_2 = \mathsf{z}$ and $\omega^1 + i\omega^2 = \omega$ in terms of z, we see that

$$X = \mathsf{Re} \int [(1 - g^2)f, i\,(1 + g^2)f, 2fg]\, dz. \tag{6.26}$$

Remark 6.4.22. Notice that multiplying f by a unit modulus constant e^{it} will produce a one-parameter family of isometric minimal surfaces parametrized by $t \in [0, 2\pi)$ The catenoid and helicoid mentioned in §2.2 are members of one such family.

The following exercises show how some well-known minimal surfaces can be obtained from (6.26) for simple choices of f and g. (Note that, more generally, $g(z)$ can be a meromorphic function whose domain has nontrivial topology, and this allows the resulting surface to have nontrivial topology itself.)

Exercises 6.4.23:

1. Identify the surfaces obtained from the Weierstrass representation using $f(z) = 1$ and $g(z) = 0$, $g(z) = z$, $g(z) = 1/z$ and $g(z) = i/z$, respectively. ◎
2. Show that the parametrization (6.26) is regular provided the poles of g coincide with (and are at most half the order of) the zeros of f.
3. Conclude that the surface obtained using $g(z) = \tan(z/2)$ and $f(z) = \cos^2(z/2)$ is regular, and identify it with one of the surfaces in question #1. (Thus, we see that the Weierstrass representation of a minimal surface is not unique.) ◎

Example 6.4.24 (Surfaces of constant mean curvature)**.** Setting $A = 0$, $B = 1$ and $C = -2H$ (where H is some constant) in the general EDS for linear Weingarten surfaces leads to a Monge-Ampère system with 2-forms $\Theta = -d\omega^3$ and $\Psi := \omega^3_1 \wedge \omega^2 + \omega^1 \wedge \omega^3_2 - 2H\omega^1 \wedge \omega^2$. As with the minimal surface system, the characteristics are complex. The decomposable 2-forms are $\Theta + i\Psi = (\pi - H\overline{\omega}) \wedge \omega$ and its complex conjugate, where π and ω are defined by (6.24).

This complex coframe, adapted as it is to the characteristics, leads to a simple basis for the EDS and a simple description for its prolongation. Namely, we adjoin a complex variable Q and define the 1-form

$$\theta := \pi - H\overline{\omega} - Q\omega.$$

The prolongation is the Pfaffian system on $\mathcal{F} \times \mathbb{C}$ generated by ω^3, θ and $\overline{\theta}$. The system 2-forms are

$$d\theta \equiv -(dQ + 2iQ\omega_2^1) \wedge \omega \mod \omega^3, \theta, \overline{\theta}$$

and its complex conjugate.

Note that Q gives the components of the traceless part of the second fundamental form. For, $\theta = 0$ implies that

$$\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} H + q_1 & -q_2 \\ -q_2 & H - q_1 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where $Q = q_1 + iq_2$. Thus, the zeros of Q are the umbilic points of the surface.

Proposition 6.4.25. *Let M be an abstract oriented surface with a Riemannian metric, and let $\mathcal{F}_{on}(M)$ be its oriented orthonormal frame bundle, with canonical forms η^1, η^2 and connection form η_2^1 (see §2.6). Endow M with the complex structure which it inherits from the metric and the orientation (see Appendix C). Suppose Q is a complex-valued function on $\mathcal{F}_{on}(M)$ that satisfies*

$$(dQ + 2iQ\eta_2^1) \wedge (\eta^1 + i\eta^2) = 0. \tag{6.27}$$

Then $\sigma = Q(\eta^1 + i\eta^2)^2$ is a well-defined holomorphic differential on M. Moreover, Q determines a local isometric embedding of M as a surface of constant mean curvature H in $\mathbb{R}^3$ which is unique up to rigid motions.

Note that σ is the *Hopf differential* associated to the Gauss map of $M \hookrightarrow \mathbb{R}^3$, which turns out to be a harmonic map into S^2 [**120**].

Proof. To see that the differential σ is well-defined, let v be the vector field tangent to the fibers of $\mathcal{F}_{on}$ that is dual to η_2^1. Then it is easy to check that $\mathcal{L}_{\mathsf{v}}\sigma = 0$. To see that it is holomorphic, let $\sigma = P(dz)^2$, where z is a local complex coordinate such that dz is a multiple of $\eta^1 + i\eta^2$. Then (6.27) implies that $\partial P/\partial \bar{z} = 0$.

As explained in §5.4, any isometric immersion of M into $\mathbb{R}^3$ will induce a immersion from $\mathcal{F}_{on}(M)$ to $\mathcal{F}_{\mathbb{E}^3}$, and the graph of this map will be a three-dimensional integral manifold of the Pfaffian system on $\mathcal{F}_{on}(M) \times \mathcal{F}_{\mathbb{E}^3}$ generated by

$$\{\omega^3, \omega^1 - \eta^1, \omega^2 - \eta^2, \omega_2^1 - \eta_2^1\}.$$

To this system we adjoin the form θ and its conjugate, where Q is now given. Our hypothesis about Q implies that this enlarged system is Frobenius. Thus, there exists a unique integral 3-fold through each point. Since the left action of $\mathcal{F}_{\mathbb{E}^3} \cong ASO(3)$ on itself covers rigid motion in $\mathbb{E}^3$, any two immersions will differ only by rigid motion. □

Corollary 6.4.26 (Bonnet). *Any constant mean curvature (CMC) surface has a circle's worth of isometric deformations as a CMC surface.*

Proof. Given $f : M \to \mathbb{R}^3$, let Q be the coefficient of the Hopf differential, as above. Then let

$$Q^t := e^{it} Q \tag{6.28}$$

for any fixed value of $t \in (0, 2\pi)$. Then Q^t also satisfies the conditions of Proposition 6.4.25. □

Moreover, if the surface isn't totally umbilic, then at most points $Q \neq 0$ and the second fundamental form is changed by (6.28). So, the surface constructed using Q^t is, in general, not congruent to the original surface by rigid motions.

Remark 6.4.27. These deformations are the CMC analogue of the one-parameter families of isometric minimal surfaces mentioned earlier. Similarly, any pseudospherical surface has a one-parameter family of non-congruent isometric deformations in $\mathbb{R}^3$. These are induced by the obvious scaling symmetry $x \mapsto \lambda x$, $y \mapsto \lambda^{-1} y$ of the sine-Gordon equation (6.20), and are known as *Lie transformations* of the given surface.

Remark 6.4.28 (Bonnet surfaces). The above deformations of CMC surfaces show that we can produce non-congruent surfaces with the same Gauss and mean curvature (in this case, H is the same constant). Bonnet discovered that, besides CMC surfaces, there is a finite-dimensional family of surfaces that admit non-congruent isometric deformations preserving H. These *Bonnet surfaces* may be characterized by the property that they admit isothermal coordinates (i.e., coordinates in which II is diagonalized, or equivalently, Q is real) and in these coordinates $1/Q$ is harmonic, i.e., $(1/Q)_{z\bar{z}} = 0$. See [**13**] and [**37**] for more information.

6.5. Integrable extensions and Bäcklund transformations

Equations which are integrable by the method of Darboux give one example where we can construct integral manifolds of an EDS by restricting to a submanifold where the system becomes Frobenius. Another example arises when the integrability condition for an overdetermined system of PDE can be reduced to finding the solution of another PDE. Then any solution of

that PDE enables us to construct (using ODE techniques) a solution of the system.

Example 6.5.1. Recall from Example 6.4.17 that the compatibility condition for the system (6.21) is that u satisfies the sine-Gordon equation. Thus, any sine-Gordon solution enables us to construct a solution to the Monge-Ampère system for pseudospherical surfaces by integrating a Frobenius system.

Example 6.5.2 (Cole-Hopf transformation). Consider the dissipative version of Burger's equation:

$$u_t = (u_x + u^2)_x. \tag{6.29}$$

Since this evolution equation is in the form of a conservation law [**127**], we can construct a potential $v(x,t)$ such that

$$v_x = u, \qquad v_t = u_x + u^2.$$

The compatibility condition for this overdetermined system is precisely that u satisfy (6.29). Hence, given a solution of (6.29), we can construct $v(x,t)$ by integration. Moreover, if $w = e^v$, then $w(x,t)$ solves the *heat equation* $w_t = w_{xx}$. In this sense, finding a solution to the heat equation is an extension of finding a solution of Burger's equation.

As in the preceding example, solutions of one PDE may give us solutions of another PDE, obtained by solving an auxiliary system of ODE. When this transformation works in both directions, the two PDE are said to be related by a *Bäcklund transformation*, which we will define later in the section. First, we formalize the relationship between the systems seen in the above examples:

Definition 6.5.3 ([**21**]). Let $\mathcal{I}$ be an EDS on a manifold Σ, let $\pi : B \to \Sigma$ be a submersion, and let $\mathcal{J}$ be an EDS defined on B. Then $\mathcal{J}$ is an *integrable extension* of $\mathcal{I}$ if there exists a Pfaffian system $J \subset T^*B$ such that

$$\mathcal{J} = \{J, \pi^*\mathcal{I}\}_{\mathsf{alg}} \tag{6.30}$$

and J is transverse to the fibers of π (i.e., a basis of J pulls back to each fiber to be a basis for the cotangent space of the fiber). The *rank* of an integrable extension is the rank of J, which is the same as the dimension of the fibers of $\pi : B \to \Sigma$.

The condition that $\mathcal{J}$, as defined by (6.30), is a differential ideal is equivalent to

$$d\theta \equiv 0 \mod J, \pi^*\mathcal{I} \tag{6.31}$$

for any $\theta \in \Gamma(J)$.

The concept of integrable extensions originated in the work of Estabrook and Wahlquist [**151, 152**], who called them *prolongation structures.* (We would also use that term, were it not for the possible confusion with the notion of prolongation for an EDS.) Integrable extensions are also closely related to *coverings* [**96**].

Example (6.5.1 continued). Let $\Sigma = \mathbb{R}^5$ and let $\mathcal{I}$ be the Monge-Ampère system for sine-Gordon given in Example 6.4.6. Let $B = \Sigma \times ASO(3)$. The equations (6.21) and (6.22) show that a Darboux framing for a pseudospherical surface associated to a sine-Gordon solution $u(x, y)$ is an integral surface for the Pfaffian system

$$J := \{\omega^1 - \cos u(dx + dy), \omega^2 + \sin u(dx - dy), \omega^1_2 - u_x dx + u_y dy,$$
$$\omega^3_1 - \sin u(dx + dy), \omega^3_2 + \cos u(dx - dy)\}.$$

Then one can check that J satisfies (6.31).

If $(B, \mathcal{J})$ is an integrable extension of rank m and $N \subset \Sigma$ is a p-dimensional integral manifold of $\mathcal{I}$, then (6.31) implies that the pullback of J to $\pi^{-1}(N)$ is Frobenius. In this way, an integral manifold of $\mathcal{I}$ gives an m-parameter family of p-dimensional integral manifolds of $\mathcal{J}$. (In Example 6.5.1, the six-parameter family of surfaces corresponding to a single solution of sine-Gordon are congruent under rigid motions of $\mathbb{E}^3$.)

Example (6.5.2 continued). To show that this example fits the definition, let $\Sigma = \mathbb{R}^4$ with coordinates x, t, u, p and let

$$\mathcal{I} = \{(du - p\,dx) \wedge dt, du \wedge dx + (dp + 2up\,dx) \wedge dt\}_{\mathsf{alg}}.$$

Let $B = \Sigma \times \mathbb{R}$, with v as new coordinate, and let $J = \{dv - u\,dx - (p + u^2)dt\}$. Again, J satisfies (6.31).

Remark 6.5.4. In general, suppose $\mathcal{I}$ is an EDS on a manifold Σ and $\Phi \in \mathcal{I}^2$ is closed. For instance, in Example 6.5.2 we have

$$\Phi = du \wedge dx + (dp + 2u\,du) \wedge dt.$$

Then on an open set U about any given point in Σ there exists a differential form ϕ such that $d\phi = \Phi$. We may define a rank one integrable extension of $\mathcal{I}$ by introducing a new coordinate y and letting the Pfaffian system J on $U \times \mathbb{R}$ be spanned by $dy - \phi$. Because such forms Φ are examples of conservation laws for an EDS [**21**], we call this construction *extension via conservation law.*

Remark 6.5.5. One could trivially satisfy Definition 6.5.3 by choosing J to be Frobenius; such an integrable extension is said to be *flat.* More generally, if the derived flag of J terminates in a Frobenius system K of rank k, then $\mathcal{J}$ defines an integrable extension when pulled back to any leaf of the foliation

dual to K. Thus, in this case $(B, \mathcal{J})$ is said to be a *k-parameter family* of integrable extensions.

Parametric families of integrable extensions are frequently encountered in the study of completely integrable PDE (i.e., "soliton" equations). We now describe the integrable extensions of the *Korteweg-de Vries* (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{6.32}$$

which were first investigated by Estabrook and Wahlquist [**151**], who encoded the equation by an EDS on $\mathbb{R}^5$.

Example 6.5.6 (KdV equation)**.** Let $\Sigma = \mathbb{R}^5$, with coordinates x, t, u, p, r. Let

$$\begin{aligned}
\Theta_1 &:= (du - p\,dx) \wedge dt,\\
\Theta_2 &:= (dp - r\,dx) \wedge dt,\\
\Theta_3 &:= (dr + 6up\,dx) \wedge dt - du \wedge dx,
\end{aligned}$$

and let $\mathcal{I} = \{\Theta_1, \Theta_2, \Theta_3\}_{\mathsf{diff}}$. Then solutions of (6.32) correspond to integral surfaces of $\mathcal{I}$ satisfying the independence condition $dx \wedge dt \neq 0$. (Note that Σ is a quotient of $J^2(\mathbb{R}^2, \mathbb{R})$, and p, r have their classical meanings as u_x and u_{xx} respectively. One could also use the EDS obtained by pulling back the standard contact system on $J^3(\mathbb{R}^2, \mathbb{R})$ to the hypersurface defined by (6.32), but the extra variables are not needed here.)

On $B = \Sigma \times \mathbb{R}^2$, with fiber coordinates (μ, y), let $J = \{d\mu, \eta\}$, where

$$\eta := dy - (y^2 + u - \mu)dx + (r + 2py + 2(u + 2\mu)(y^2 + u - \mu))dt,$$

and let $\mathcal{J} = \{J, \mathcal{I}\}_{\mathsf{alg}}$.

Exercise 6.5.7: Verify that $\mathcal{J}$ is an integrable extension of $\mathcal{I}$.

Since J contains a rank-one Frobenius system, we have a one-parameter family of extensions, with μ as a parameter. In fact, along an integral surface of $\mathcal{J}$ that satisfies the independence condition, $y(x,t)$ satisfies

$$y_t + y_{xxx} + 6(\mu - y^2)y_x = 0 \tag{6.33}$$

for some constant μ. The family (6.33) of PDE may be said to be extensions of KdV. When $\mu = 0$, (6.33) is a version of another well-known PDE, the *modified KdV equation*, and the passage from $u(x,t)$ to $y(x,t)$ is known as the *Miura transformation* [**132**].

The integrable extensions of the KdV equation were classified by Estabrook and Wahlquist [**151**]. Using a local normal form for the extension, they showed that each extension corresponds to a finite-dimensional representation of a certain infinite-dimensional Lie algebra, known as the *KdV*

prolongation algebra. Moreover, this algebra has a one-parameter family of finite-dimensional quotients, each of which is isomorphic to a semidirect product of $\mathfrak{sl}(2,\mathbb{R})$ with $\mathbb{R}^5$ [**138**]. Representations of the latter algebra allow us to construct one-parameter families of KdV extensions, the most well-known of which is the AKNS system [**132**]:

$$\psi_x = \begin{pmatrix} \lambda & u \\ -1 & -\lambda \end{pmatrix} \psi, \qquad \psi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \psi, \tag{6.34}$$

where $\psi(x,t) \in \mathbb{R}^2$ and $A = -4\lambda^3 - 2u\lambda - u_x$, $B = -u_{xx} - 2\lambda u_x - 4\lambda^2 u - 2u^2$ and $C = 4\lambda^2 + 2u$. This system, which is central to the solution of (6.32) by inverse scattering, is sometimes regarded as defining a connection on the trivial bundle $M \times \mathbb{R}^2$, for which (6.32) is the "zero-curvature condition".

The over-determined system (6.34) of PDE may be encoded by a Pfaffian system $\hat{\mathcal{J}}$ on $\hat{B} = M \times \mathbb{R}^3$, giving an integrable extension of rank three. Moreover, on the subset of $\hat{B}$ where $\psi_2 \neq 0$ we may define a bundle map $f : \hat{B} \to B$ by letting $\mu = \lambda^2$ and $y = \psi_1/\psi_2 + \lambda$, so that $f^*\mathcal{J} \subset \hat{\mathcal{J}}$.

Exercise 6.5.8: As the above example suggests, an integrable extension may sometimes be factored as a sequence $(\hat{B}, \hat{\mathcal{J}}) \to (B, \mathcal{J}) \to (M, \mathcal{I})$. Given $(\hat{B}, \hat{\mathcal{J}}) \to (M, \mathcal{I})$, determine conditions under which such a factorization exists. ◎

Bäcklund transformations. As described earlier, Bäcklund transformations are a particular form of integrable extension which allows a two-way transformation of solutions of a PDE to solutions of another PDE or the same PDE. One of the most well-known of these is the following transformation, which transforms "old" solutions of the sine-Gordon equation (6.20) to "new" solutions of the same PDE:

Example 6.5.9.

$$\begin{aligned} u_x - \bar{u}_x &= \lambda \sin(u + \bar{u}), \\ u_y + \bar{u}_y &= \frac{1}{\lambda} \sin(u - \bar{u}). \end{aligned} \tag{6.35}$$

If $u(x,y)$, $\bar{u}(x,y)$ are smooth functions that satisfy this system for some $\lambda \neq 0$, then differentiating and equating mixed partials implies that u and $\bar{u}$ must be solutions of sine-Gordon. Conversely, given a solution $u(x,y)$ of (6.20), we can determine $\bar{u}_x$ and $\bar{u}_y$ from (6.35), and integrate to get the new solution $\bar{u}(x,y)$ (since we are guaranteed that mixed partials commute). For example, the traveling wave solution (6.23) can be produced in this way, by starting with the identically zero solution of (6.20).

Bäcklund transformations occur frequently in the differential geometry of surfaces [**133**], and for this reason it is useful to have a coordinate-free definition for them:

Definition 6.5.10. Let Σ_1, Σ_2 carry EDS's $\mathcal{I}_1, \mathcal{I}_2$ respectively, and let $B \subset \Sigma_1 \times \Sigma_2$ be a submanifold such that the projections $\pi_k : B \to \Sigma_k$ give B the structure of a double fibration. Then $(B, \mathfrak{J})$ defines a *Bäcklund transformation* between $\mathcal{I}_1$ and $\mathcal{I}_2$ if $\mathfrak{J}$ is an EDS on B that is an integrable extension of both $\mathcal{I}_1$ and $\mathcal{I}_2$.

(This definition is a generalization of the notion of a *Bäcklund map* [**134**].)

We will often define Bäcklund transformations by giving a Pfaffian system $J \subset T^*B$ such that

$$\{J, \pi_1^*\mathcal{I}_1\}_{\mathsf{alg}} = \{J, \pi_2^*\mathcal{I}_2\}_{\mathsf{alg}}, \tag{6.36a}$$

$$\mathfrak{J} := \{J, \pi_k^*\mathcal{I}_k\} \text{ is a differential ideal.} \tag{6.36b}$$

Note that J need not be transverse to the fibers of π_1 or π_2.

Example (6.5.9 continued)**.** Let Σ_1 and Σ_2 be two copies of $J^1(\mathbb{R}^2, \mathbb{R})$, where we denote the forms and coordinates on Σ_2 with bars. For $i = 1, 2$ let the EDS $\mathcal{I}_i$ on Σ_i be a copy of the Monge-Ampère system for the sine-Gordon equation, described in Example 6.4.6. For any $\lambda \neq 0$, let $B_\lambda \subset \Sigma_1 \times \Sigma_2$ be defined by $\bar{x} = x$, $\bar{y} = y$ and

$$p - \bar{p} = \lambda \sin(u + \bar{u}),$$
$$q + \bar{q} = \frac{1}{\lambda} \sin(u - \bar{u}).$$

Let θ be the standard contact form on $J^1(\mathbb{R}^2, \mathbb{R})$. On B_λ, let $J = \{\pi_1^*\theta, \pi_2^*\bar{\theta}\}$. Then it is easy to check that (6.36) is satisfied.

Example (6.5.6 continued)**.** Let Σ_1, Σ_2 be two copies of $\mathbb{R}^5$, each carrying a copy of the KdV exterior differential system. (Again, we use barred coordinates on Σ_2.) Define $B = \Sigma_1 \times \mathbb{R}^2$ as before, and define a submersion $B \to \Sigma_2$ by

$$\begin{aligned}
\bar{x} &= x, \\
\bar{y} &= y, \\
\bar{u} &= 2(\mu - y^2) - u, \\
\bar{p} &= 4y(\mu - u - y^2) - p, \\
\bar{r} &= -4(u + y^2 - \mu)(u + 3y^2 - \mu) - 4yp - r.
\end{aligned}$$

Then J, as defined before, gives a Bäcklund transformation between the KdV equation and itself. In fact, one can check that

$$\left.\begin{aligned}
\pi_2^*\Theta_1 &\equiv -\pi_1^*\Theta_1 \\
\pi_2^*\Theta_2 &\equiv -\pi_1^*\Theta_2 - 4y\pi_1^*\Theta_1 \\
\pi_2^*\Theta_3 &\equiv -\pi_1^*\Theta_3 - 4y\pi_1^*\Theta_2 - 8(u + 2y^2 - \mu)\pi_1^*\Theta_1
\end{aligned}\right\} \quad \mod J.$$

The presence of an arbitrary parameter in the Bäcklund transformation for KdV is central to its complete integrability, both in the sense of inverse scattering and in the Hamiltonian sense. For example, it is shown in [**150**] that expanding $u+\bar{u}$ as a power series in μ^{-1} reveals the existence of infinitely many independent conservation laws for (6.32).

Of course, it is not necessary that a Bäcklund transformation take solutions to solutions for the same PDE:

Example 6.5.11. Let $\Sigma_1 \subset J^2(\mathbb{R}^2, \mathbb{R})$ be the submanifold defined by Liouville's equation $z_{xy} = e^z$, with the standard contact system $\mathcal{I}_1$ generated by $\theta_0, \theta_1, \theta_2$ defined in Example 6.3.6. Use the classical notation p, q, r, s, t for the jet coordinates restricted to Σ_1. Then (6.14) implies that $(r - \frac{1}{2}p^2)_y = 0$ and $(t - \frac{1}{2}q^2)_x = 0$ for any solution. Thus, on $B = \Sigma_1 \times \mathbb{R}$, with new coordinate v, we may define an extension by letting $J = \{\theta_0, \theta_1, \theta_2, \eta\}$, where

$$\eta := dv - (r - \tfrac{1}{2}p^2)dx - (t - \tfrac{1}{2}q^2)dy.$$

Because $d\eta \equiv 0 \mod \mathcal{I}_1$, $\mathfrak{J} = \{J, \mathcal{I}_1\}_{\mathsf{alg}}$ is an integrable extension of $\mathcal{I}_1$.

Furthermore, let $\Sigma_2 = J^2(\mathbb{R}^2, \mathbb{R})$ with coordinates $\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}$, and let $\mathcal{I}_2$ be the standard Monge-Ampère system for the wave equation $\bar{u}_{\bar{x}\bar{y}} = 0$. Define a submersion $\pi_2 : B \to \Sigma_2$ by

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = v, \quad \bar{p} = r - \tfrac{1}{2}p^2, \quad \bar{q} = t - \tfrac{1}{2}q^2.$$

Then $\mathfrak{J}$ is an integrable extension of $\mathcal{I}_2$ (with the role of the transverse Pfaffian system played by $\{\theta_0, \theta_1, \theta_2\}$). Thus, $(B, \mathfrak{J})$ defines a Bäcklund transformation between Liouville's equation and the wave equation.

Example 6.5.12. Let Σ_1 and Σ_2 be two copies of the frame bundle $\mathcal{F}$ of Euclidean space $\mathbb{E}^3$, each carrying a copy $\mathcal{I}_k$ of the Monge-Ampère system for pseudospherical surfaces, defined in Example 6.4.17. (The frame vectors and canonical forms on Σ_2 will be distinguished from those on Σ_1 by bars.)

The Bäcklund transformation for pseudospherical surfaces arises naturally in a classical context, the study of *line congruences* in Euclidean space. By definition, a line congruence is a two-parameter family of lines; associated to the congruence are two *focal surfaces* to which each line in the family is tangent. Leaving degeneracies aside, the lines locally give a 1-to-1 correspondence between points on the focal surfaces.

Theorem (Bäcklund). *If the distance λ between corresponding points on the focal surfaces and the angle ψ between the surface normal at corresponding points are both constant, then the two surfaces have constant Gauss curvature equal to $-\sin^2\psi/\lambda^2$.*

(We will take $\lambda = \sin\psi$, so that $K = -1$.)

The starting point for a proof of Bäcklund's Theorem (cf. [**39**]) is adapting frames (X, e_1, e_2, e_3) and $(\bar{X}, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ along the two surfaces so that $e_1 = \bar{e}_1$ is tangent to the line connecting corresponding points X and $\bar{X}$. For any fixed λ, the graph of the Bäcklund transformation is a 6-dimensional submanifold of $\mathcal{F} \times \bar{\mathcal{F}}$ defined by

$$\begin{aligned} \bar{X} &= X + \lambda e_1, \\ \bar{e}_1 &= e_1, \\ \bar{e}_2 &= e_2 \cos\psi + e_3 \sin\psi, \\ \bar{e}_3 &= e_3 \cos\psi - e_2 \sin\psi. \end{aligned} \tag{6.37}$$

However, we will regard (6.37) as defining a 7-dimensional submanifold $B \subset \mathcal{F} \times \bar{\mathcal{F}} \times \mathbb{R}$, with λ as the extra coordinate. On B, let $J = \{\omega^3, \bar{\omega}^3, d\lambda\}$. Then (6.36) is satisfied, and Bäcklund's Theorem follows from the fact that

$$\{\omega^3_1 \wedge \omega^3_2 + \omega^1 \wedge \omega^2, \bar{\omega}^3_1 \wedge \bar{\omega}^3_2 + \bar{\omega}^1 \wedge \bar{\omega}^2\} \equiv \{d\omega^3, d\bar{\omega}^3\} \bmod J.$$

Exercises 6.5.13:

1. Differentiate (6.37) to obtain the following relationships between the pullbacks to B of the canonical forms:

$$\bar{\omega}^1 = \omega^1 + d\lambda, \qquad \bar{\omega}^2_3 = \omega^2_3 - d\psi,$$
$$\begin{pmatrix} \bar{\omega}^2 \\ \bar{\omega}^3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \omega^2 - \lambda\omega^1_2 \\ \omega^3 - \lambda\omega^1_3 \end{pmatrix},$$
$$\begin{pmatrix} \bar{\omega}^1_2 \\ \bar{\omega}^1_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \omega^1_2 \\ \omega^1_3 \end{pmatrix}.$$

Then, use these relationships to verify (6.36).

2. Show that asymptotic lines are taken to asymptotic lines under the Bäcklund transformation. ◎

3. Show that if two pseudospherical surfaces M_1, M_2 are related by a Bäcklund transformation and both are generated by solutions $u_i(x, y)$ of the sine-Gordon equation, then u_1 and u_2 are related by a Bäcklund transformation for the same λ-value.

4. ([**158**]) Consider the system defined by

$$u_x = \frac{u}{x+y} + \sqrt{2w_x}, \qquad u_y = \frac{u}{x+y} + \sqrt{2w_y}. \tag{6.38}$$

Show that this defines a Bäcklund transformation between the wave equation $u_{xy} = 0$ and

$$w_{xy} + 2\frac{\sqrt{w_x}\sqrt{w_y}}{x+y} = 0. \tag{6.39}$$

5. Explain why the Cole-Hopf transformation (Example 6.5.2) is not a Bäcklund transformation. ◎

6. Show that existence of a Bäcklund transformation between two systems is an equivalence relation on exterior differential systems. ◎

The appearance of Liouville's equation in both Example 6.3.6 and 6.5.11 leads one to suspect a connection between Bäcklund transformations and Darboux integrability. The following theorem[2] bears this out:

Theorem 6.5.14. *Let $\mathcal{I}_1$ be a hyperbolic EDS of class k, with independence condition, on Σ_1^{k+4}, and let $\mathcal{I}_2$ be the Monge-Ampère system on $\Sigma_2 = \mathbb{R}^5$ that encodes the wave equation $u_{xy} = 0$, with independence condition $dx \wedge dy \neq 0$. If $\mathcal{I}_1$ is Darboux-integrable, then around each point $s \in \Sigma_1$ there exist an open set U, an open set $V \subset \Sigma_2$, and a Bäcklund transformation $(B, \mathcal{J})$ between $\mathcal{I}_1|_U$ and $\mathcal{I}_2|_V$ such that $\dim B = k+5$ and the pullbacks of the independence conditions are equivalent modulo $\mathcal{J}$. Conversely, if such a Bäcklund transformation exists, then $\mathcal{I}_1$ is Darboux-integrable after at most one prolongation.*

Proof. First, suppose that $\mathcal{I}_1$ is Darboux-integrable and let $\mathcal{M}_1, \mathcal{M}_2$ be its characteristic systems. By the Frobenius Theorem, on an open set U containing s there exist functions x, y, p, q such that $\Delta_1 = \{dx, dp\} \subset \mathcal{M}_1$ and $\Delta_2 = \{dy, dq\} \subset \mathcal{M}_2$. Each Δ_i is independent of $\mathcal{I}_1$, so

$$\mathcal{I}_1|_U = \{\theta_1, \dots, \theta_k, dp \wedge dx, dq \wedge dy\}_{\mathsf{alg}}. \tag{6.40}$$

Moreover, we may choose x, y, p, q such that $dx \wedge dy \neq 0$ is equivalent, modulo $\mathcal{I}_1$, to the independence condition.

The graph of the four functions x, y, p, q is a submanifold $U' \subset U \times \mathbb{R}^4$. Let $B = U' \times \mathbb{R}$, with u as the additional coordinate, and let W be the image of $U' \times \mathbb{R}$ under the natural projection onto $\mathbb{R}^5$ with coordinates (x, y, p, q, u). On $\mathbb{R}^5$, let

$$\mathcal{I}_2 = \{du - p\,dx - q\,dy, dp \wedge dx, dq \wedge dy\}_{\mathsf{alg}}. \tag{6.41}$$

On B, let the Pfaffian system J be spanned by the pullbacks of the 1-forms of $\mathcal{I}_1$ and the pullback from W of $\theta_0 = du - p\,dx - q\,dy$. Because $d\theta_0 \equiv 0 \mod \mathcal{I}_1$, we see that $\mathcal{J} = \{J, \pi_1^*\mathcal{I}_1\}$ is a differential ideal. Comparing (6.40) and (6.41) shows that $\pi_1^*\mathcal{I}_1 \subset \{J, \pi_2^*\mathcal{I}_2\}_{\mathsf{alg}}$ and $\pi_2^*\mathcal{I}_2 \subset \{J, \pi_1^*\mathcal{I}_1\}_{\mathsf{alg}}$. Therefore, we have a Bäcklund transformation.

Next, suppose that $(B, \mathcal{J})$ defines a (local) Bäcklund transformation of the specified dimension between $\mathcal{I}_1|_U$ and $\mathcal{I}_2|_W$. On a possibly smaller set $U \subset \Sigma_1$ there exists a coframe $\theta_1, \dots, \theta_k, \omega_1, \omega_2, \varphi_1, \varphi_2$ such that

$$\mathcal{I}_1 = \{\theta_1, \dots, \theta_k, \varphi_1 \wedge \omega_1, \varphi_2 \wedge \omega_2\}_{\mathsf{alg}}$$

[2]We believe this theorem is due to Goursat; however, we have been unable to locate it in the literature.

and such that $\omega_1 \wedge \omega_2 \neq 0$ is the independence condition. Because of (6.36a),

$$\mathcal{J} = \{\theta_0, \theta_1, \dots, \theta_k, dp \wedge dx, dq \wedge dy\}_{\mathsf{alg}}.$$

(For simplicity, pullbacks are omitted here.) Thus, $\mathcal{J}$ is a hyperbolic EDS of class $k+1$. Its prolongation is

$$\mathcal{J}' = \{\theta_0, \theta_1, \dots, \theta_k, dp - r\,dx, dq - t\,dy\}_{\mathsf{diff}},$$

defined on $P = B \times \mathbb{R}^2$. The characteristic systems of $\mathcal{J}'$ are

$$\begin{aligned}
\mathcal{K}_1' &= \{\theta_0, \theta_1, \dots \theta_k, dp, dr, dx\},\\
\mathcal{K}_2' &= \{\theta_0, \theta_1, \dots, \theta_k, dq, dt, dy\}.
\end{aligned}$$

In particular, each $\mathcal{K}_i'$ contains a Frobenius subsystem Δ_i' of rank three.

On the other hand, (6.36a) also implies that

$$\mathcal{J} = \{\theta_0, \theta_1, \dots, \theta_k, \varphi_1 \wedge \omega_1, \varphi_2 \wedge \omega_2\}_{\mathsf{alg}},$$

and hence

$$\mathcal{J}' = \{\theta_0, \theta_1, \dots, \theta_k, \varphi_1 - p_1\omega_1, \varphi_2 - p_2\omega_2\}_{\mathsf{diff}}$$

for some functions p_1, p_2 on Q. Let $\mathcal{I}'$ be the prolongation of $\mathcal{I}$, defined on a bundle Q^{k+6} over Σ_1 (restricted to W if necessary). Then $\mathcal{J}'$ in an integrable extension of $\mathcal{I}'$. We summarize the relationships between these systems in the following diagram:

$$\begin{tikzcd}
& (P,\mathcal{J}') \arrow[dl] \arrow[dr] & & \\
(Q,\mathcal{I}') \arrow[dr] & & (B,\mathcal{J}) \arrow[dl] \arrow[dr] & \\
& (M_1,\mathcal{I}_1) & & (\mathbb{R}^5,\mathcal{I}_2)
\end{tikzcd}$$

Let $\mathcal{M}_1', \mathcal{M}_2'$ be the characteristic systems of $\mathcal{I}'$. Without loss of generality, we may assume that $\rho^*\mathcal{M}_i' \subset \mathcal{K}_i'$ for $i = 1, 2$. Since $\rho^*\mathcal{M}_i'$ is codimension one in $\mathcal{K}_i'$, it has at least a two-dimensional intersection with Δ_i'. But any rank two system contained in a three-dimensional Frobenius system is itself Frobenius. Therefore, $\mathcal{I}'$ is Darboux-integrable. $\square$

Exercises 6.5.15:

1. The above theorem implies that (6.39) is Darboux-integrable; verify this.

2. Goursat ([**63**], §173) remarks that (6.39) is one of the few known Darboux-integrable equations with a finite-dimensional symmetry group. Verify that the transformations

$$w \mapsto Aw + B, \qquad x \mapsto \frac{ax+b}{cx+d}, \qquad y \mapsto -\frac{ay-b}{cy-d} \tag{6.42}$$

are symmetries of this equation.

Remark 6.5.16. If we only knew the Bäcklund transformation (6.35) for the sine-Gordon equation for the value $\lambda = 1$, we could still "lift" the Lie symmetry to the total space B^6 of the transformation in a way that enables us to recover the entire family. In the same way, the symmetries (6.42) may be used to embed (6.38) in a one-parameter family of Bäcklund transformation for (6.39); see [**40**] for details.

Chapter 7

Cartan-Kähler III: The General Case

In this chapter we will discuss the Cartan-Kähler Theorem, which guarantees the existence of integral manifolds for arbitrary exterior differential systems in involution. This theorem is a generalization of the Cauchy-Kowalevski Theorem (see Appendix D), which gives conditions under which an analytic system of partial differential equations has an analytic solution (defined in the neighborhood of a given point) satisfying a Cauchy problem, i.e., initial data for the solution specified along a hypersurface in the domain. Similarly, the "initial data" for the Cartan-Kähler Theorem is an integral manifold of dimension n, which we want to extend to an integral manifold of dimension $n+1$. In Cauchy-Kowalevski, the equations are assumed to be of a special form, which has the feature that no conflicts arise when one differentiates them and equates mixed partials. The condition of involutivity is a generalization of this, guaranteeing that no new integrability conditions arise when one looks at the equations that higher jets of solutions must satisfy.

The reader may wonder why one bothers to consider any exterior differential systems other than linear Pfaffian systems. For, as remarked in Chapter 5, the prolongation of any exterior differential system is a linear Pfaffian system, so theoretically it is sufficient to work with such systems. However, in practice it is generally better to work on the smallest space possible. In fact, certain spectacular successes of the EDS machinery were obtained by cleverly rephrasing systems that were naïvely expressed as linear Pfaffian systems as systems involving generators of higher degree on a smaller manifold. One elementary example of this is the study of linear

Weingarten surfaces (see §6.4); a more complex example is Bryant's proof of the existence of manifolds with holonomy G_2 [**17**].

We begin this chapter with a more detailed study of the space of integral elements of an EDS. Before proving the full Cartan-Kähler Theorem, in §7.2 we give an example that shows how one can use the Cauchy-Kowalevski Theorem to construct triply orthogonal systems of surfaces. (This also serves as a model for the proof in §7.3.) In §7.4 we discuss Cartan's Test, a procedure by which one can test for involution, and which was already described in Chapter 5 for the special case of linear Pfaffian systems. Then in §7.5 we give a few more examples that illustrate how one applies Cartan's Test in the non-Pfaffian case.

7.1. Integral elements and polar spaces

Suppose $\mathcal{I}$ is an exterior differential system on Σ; we will assume that $\mathcal{I}$ contains no 0-forms (otherwise, we could restrict to subsets of Σ on which the 0-forms vanish). Recall from §1.9 that $\mathcal{V}_n(\mathcal{I})_p \subset G(n, T_p\Sigma)$ denotes the space of n-dimensional integral elements in $T_p\Sigma$ and $\mathcal{V}_n = \mathcal{V}_n(\mathcal{I}) \subset G_n(T\Sigma)$ the space of all n-dimensional integral elements. In this chapter we will obtain a criterion that guarantees that a given $E \in \mathcal{V}_n(\mathcal{I})_p$ is tangent to an integral manifold. We think of this as "extending" (p, E) to an integral manifold.

Coordinates on $G_n(T\Sigma)$. To study the equations that define $\mathcal{V}_n$, we use local coordinates on the Grassmann bundle $G_n(T\Sigma)$. Given $E \in G_n(T_p\Sigma)$, there are coordinates $x^1, \dots, x^n$ and $y^1, \dots, y^s$ on Σ near p such that E is spanned by the vectors $\partial/\partial x^i$. By continuity, there is a neighborhood of E in $G_n(T\Sigma)$ consisting of n-planes $\widetilde{E}$ such that $dx^1 \wedge \dots \wedge dx^n|_{\widetilde{E}} \neq 0$. For each $\widetilde{E}$, there are numbers p^a_i such that $dy^a|_{\widetilde{E}} = p^a_i dx^i|_{\widetilde{E}}$; these p^a_i, along with the x's and y's, form a local coordinate system on $G_n(T\Sigma)$.

Recall from Exercise 1.9.4 that $E \in \mathcal{V}_n(\mathcal{I})$ if and only if every $\psi \in \mathcal{I}^n$ vanishes on E. Each such ψ has some expression

$$\psi = \sum_{I,J} f_{IJ}\, dy^I \wedge dx^J,$$

where I and J are multi-indices with components in increasing order, such that $|I| + |J| = n$, and the f_{IJ} are smooth functions on Σ. Then $\psi|_{\widetilde{E}} = F_\psi dx^1 \wedge \dots \wedge dx^n|_{\widetilde{E}}$, where the F_ψ are polynomials in the p^a_i, given by

$$F = \sum_{I,J,L} f_{I,J}(x,y) p^{i_1}_{l_1} \dots p^{i_k}_{l_k} dx^L \wedge dx^J,$$

with $I = (i_1, \ldots, i_k)$ and $L = (l_1, \ldots, l_k)$ in increasing order. Thus, $\mathcal{V}_n(\mathcal{I})$ is defined by equations that are polynomial in the p_i^a, with coefficients that are smooth functions on Σ.

Smooth points of $\mathcal{V}_n(\mathcal{I})$. To show existence of integral manifolds with tangent space E at $p \in \Sigma$, we will need to study integral elements near E. We will generally restrict to integral elements that are smooth points of $\mathcal{V}_n(\mathcal{I})$, so we make the following definitions:

Definition 7.1.1. We say that k is the *codimension* of $\mathcal{V}_n(\mathcal{I})$ at E if k is the maximum number of smooth functions F_ψ on $G_n(T\Sigma)$ that vanish on $\mathcal{V}_n(\mathcal{I})$ and have linearly independent differentials at E.

Definition 7.1.2. An integral element $E \in \mathcal{V}_n(\mathcal{I})$ is *Kähler-ordinary* if $\mathcal{V}_n(\mathcal{I})$ is a smooth submanifold of $G_n(T\Sigma)$ near E.

If the codimension is constant on a neighborhood of E, then E is Kähler-ordinary. If not, then by continuity of the coefficients in those polynomial equations, there will be a neighborhood of E in which $\mathcal{V}_n$ has codimension at least k. (In other words, codimension is a lower semicontinuous function on $\mathcal{V}_n$.) Since the codimension is bounded above, Kähler-ordinary elements form an open dense set in $\mathcal{V}_n$.

Example 7.1.3 (*Linear Pfaffian systems*)**.** Suppose linearly independent 1-forms θ^a, $1 \le a \le s$, generate a Pfaffian system on Σ with independence condition $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$, and satisfy structure equations

$$d\theta^a \equiv \pi_i^a \wedge \omega^i \mod \theta^1, \ldots, \theta^s.$$

Complete $\{\theta^a, \omega^i\}$ to a coframing with forms π^ϵ, $1 \le \epsilon \le r$. Then $\pi_i^a = A_{i\epsilon}^a \pi^\epsilon + C_{ij}^a \omega^j$ for some functions $A_{i\epsilon}^a, C_{ij}^a$ on Σ.

On any n-plane E satisfying the independence condition, $\pi^\epsilon = p_i^\epsilon \omega^i$. (As above, the p_i^ϵ form part of a local coordinate system on $G_n(T\Sigma)$.) If E is an integral element, then the θ's must vanish on E and

$$A_{i\epsilon}^a p_j^\epsilon - A_{j\epsilon}^a p_i^\epsilon + C_{ij}^a - C_{ji}^a = 0. \tag{7.1}$$

Since these equations are linear in the p_i^ϵ, $\mathcal{V}_n(\mathcal{I})$ is a smooth submanifold wherever the rank of (7.1) is locally constant. Hence, if one point in the fiber of $\mathcal{V}_n$ is smooth, so are all the other points in the fiber.

Non-smooth points of $\mathcal{V}_n(\mathcal{I})$ can occur in the fibers over points in Σ where the generators of $\mathcal{I}$ vanish or become linearly dependent. However, as Exercise 7.1.5 shows, this is not the only way in which singular points arise.

Example 7.1.4. On $\Sigma = \mathbb{R}^2$, let $\theta = y^2\,dx - x\,dy$ and let $\mathcal{I} = \{\theta\}_{\mathsf{diff}}$. Over every point of Σ except the origin, the fiber of $\mathcal{V}_1(\mathcal{I})$ is a single point, and $\mathcal{V}_1$ is smooth. Since θ vanishes at the origin, the fiber there is $G_1(\mathbb{R}^2) = \mathbb{RP}^1$. We can introduce a local fiber coordinate on $G_1(T\Sigma)$ such that $dy - p\,dx|_E = 0$. Then $\mathcal{V}_1$ is defined by $y^2 - xp = 0$, and the integral 1-plane with $p = 0$ at the origin is a singular point of $\mathcal{V}_1$. Although $\mathcal{V}_1$ is smooth at all the other points in the fiber above the origin, only the directions along the coordinate axes are tangent to integral curves of $\mathcal{I}$.

Exercise 7.1.5 (*A degenerate 2-form*)**:** On $\Sigma = \mathbb{R}^m$, let $\omega^1, \omega^2 \in \Omega^1(\Sigma)$ be pointwise independent and let $\mathcal{I} = \{\omega^1 \wedge \omega^2\}_{\mathsf{alg}}$. Complete ω^1, ω^2 to a coframing $\omega^1, \dots, \omega^m$ with dual framing $e_1, \dots, e_m$. Fix a basepoint p and show that $e_3 \wedge e_4$ is a singular point of $\mathcal{V}_2(\mathcal{I})_p$. Determine all singular points. ◎

More generally, we say that a form $\varphi \in \Omega^k(\Sigma)$, $k \geq 2$, is *degenerate* at p if

$$\operatorname{Ann}(\varphi)_p := \{v \in T_p\Sigma \mid v \lrcorner \phi = 0\}$$

is nonzero. For $\mathcal{I} = \{\phi\}_{\mathsf{alg}}$, there will be singular points of $\mathcal{V}_k(\mathcal{I})$ in the fiber over every point, consisting of those $E \in \mathcal{V}_k$ that, as subspaces of $T_p\Sigma$, intersect $\operatorname{Ann}(\varphi)_p$ non-transversely.

Polar spaces. The infinitesimal analogue of the Cauchy problem described at the beginning of the chapter is the extension of an n-dimensional integral element $E \subset T_p\Sigma$ to an $(n+1)$-dimensional integral element $E^+ \subset T_p\Sigma$. The space of all extensions is a (possibly empty) projective space $\mathbb{P}(H(E)/E)$ where $H(E) \subseteq T_p\Sigma$ is the *polar space* of E, defined as follows:

Definition 7.1.6. Let $e_1, \dots, e_n$ be any basis for the integral element $E \subset T_p\Sigma$. The polar space of E is

$$H(E) := \{v \in T_p\Sigma \mid \psi(v, e_1, \dots, e_n) = 0\ \forall \psi \in \mathcal{I}^{n+1}\}.$$

Exercise 7.1.7 (*Properties of Polar Spaces*)**:** Let $E \in \mathcal{V}_n(\mathcal{I})_p$, $E^+ \in G_{n+1}(T_p\Sigma)$ and $E \subset E^+$. Then
1. $E \subseteq H(E)$
2. $E^+ \subseteq H(E)$ if and only if $E^+ \in \mathcal{V}_{n+1}(\mathcal{I})$.
3. If $v \in H(E)$ then $(v \lrcorner \psi)|_E = 0$ for all $\psi \in \mathcal{I}$. ◎
4. If $E^+ \in \mathcal{V}_{n+1}(\mathcal{I})$, then $H(E^+) \subseteq H(E)$.
5. If $\{\psi^\alpha\}$ is a set of algebraic generators of $\mathcal{I}$, then $v \in H(E)$ if and only if $(v \lrcorner \psi^\alpha)|_E = 0$ for all generators ψ^α of degree at most $n+1$.

The following examples show how polar spaces are calculated.

Example 7.1.8. Suppose the forms θ_1, θ_2, ω^1, ω^2 and π form a coframe on a manifold Σ^5 such that

$$\left.\begin{aligned} d\theta_1 &\equiv \omega^1 \wedge \pi \\ d\theta_2 &\equiv \omega^2 \wedge \pi \end{aligned}\right\} \mod \theta_1, \theta_2 \tag{7.2}$$

at some basepoint $p \in \Sigma$. Let $\mathcal{I} = \{\theta_1, \theta_2\}_{\mathsf{diff}}$. Let $E = \{\theta_1, \theta_2, \omega^2, \pi\}^\perp \subset T_p\Sigma$—that is, the 1-plane in $T_p\Sigma$ annihilated by θ_1, θ_2, ω^2 and π—and let $e \in E$ satisfy $\omega^1(e) = 1$. (For example, consider the special case where $\Sigma = \mathbb{R}^5$ with coordinates (x, y, z, w, q), and $\theta_1 = dz - q dx$, $\theta_2 = dw - q dy$, $\pi = dq$, $\omega^1 = dx$, $\omega^2 = dy$. Then $e = \frac{\partial}{\partial x} + q\frac{\partial}{\partial z}$.)

The 2-forms of $\mathcal{I}$ are spanned by $\omega^1 \wedge \pi$, $\omega^2 \wedge \pi$ and the wedge products of θ_1 and θ_2 with all 1-forms. Suppose $v \in H(E)$. Now, $\theta_1 \wedge \phi\,(v, e) = 0$ for all 1-forms ϕ if and only if $\theta_1(v) = 0$. Similarly, we must have $\theta_2(v) = 0$. Then

$$\begin{aligned} \omega^1 \wedge \pi\,(v, e) &= -\pi(v), \\ \omega^2 \wedge \pi\,(v, e) &= 0 \end{aligned}$$

show that $H(E) = \{\theta_1, \theta_2, \pi\}^\perp$. Hence E is contained in the unique integral 2-plane $E^+ = H(E)$.

Now consider $\widetilde{E} = \{\theta_1, \theta_2, \omega^1, \omega^2\}^\perp \subset T_p\Sigma$, and let $\tilde{e} \in \widetilde{E}$ satisfy $\pi(\tilde{e}) = 1$. (In the special case above, this corresponds to $\tilde{e} = \frac{\partial}{\partial p}$.) For the same reasons, if $v \in H(\widetilde{E})$, then $\theta_1(v) = 0$ and $\theta_2(v) = 0$. Then

$$\begin{aligned} \omega^1 \wedge \pi\,(v, \tilde{e}) &= \omega^1(v), \\ \omega^2 \wedge \pi\,(v, \tilde{e}) &= \omega^2(v) \end{aligned}$$

show that $H(\widetilde{E}) = \widetilde{E}$, so $\widetilde{E}$ is contained in no higher-dimensional integral element.

Example 7.1.9. Suppose the forms θ_1, θ_2, ω^1, ω^2, π_1 and π_2 form a coframe on a manifold Σ^6 such that

$$\left.\begin{aligned} d\theta_1 &\equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \\ d\theta_2 &\equiv \omega^1 \wedge \pi_2 \end{aligned}\right\} \mod \theta_1, \theta_2$$

at some basepoint $p \in \Sigma$. Let $\mathcal{I} = \{\theta_1, \theta_2\}_{\mathsf{diff}}$, let $E = \{\theta_1, \theta_2, \omega^2, \pi_1, \pi_2\}^\perp \subset T_p\Sigma$ and let $e \in E$ satisfy $\omega^1(e) = 1$. By polar space property 7.1.7.5 above, $v \in H(E)$ if and only $v \lrcorner \theta_1 = 0$, $v \lrcorner \theta_2 = 0$, and v is annihilated by the interior products of e with all generator 2-forms. Since these are

$$\begin{aligned} e \lrcorner\,(\omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2) &= \pi_1, \\ e \lrcorner\,(\omega^1 \wedge \pi_2) &= \pi_2, \end{aligned}$$

we have $H(E) = \{\theta_1, \theta_2, \pi_1, \pi_2\}^\perp$.

Now let $\widetilde{E} = \{\theta_1, \theta_2, \omega^1, \pi_1, \pi_2\}^\perp$, and let $\tilde{e} \in \widetilde{E}$ satisfy $\omega^2(\tilde{e}) = 1$. Then

$$\tilde{e} \lrcorner (\omega^1 \wedge \pi_1 + \omega^2 \wedge \pi^2) = \pi_2,$$
$$\tilde{e} \lrcorner (\omega^1 \wedge \pi_2) = 0$$

shows that $H(\widetilde{E}) = \{\theta_1, \theta_2, \pi_2\}^\perp$. Note that $H(\widetilde{E})$ is not itself an integral 3-plane, since $(\omega^1 \wedge \pi_1 + \omega^2 \wedge \pi^2)\mid_{H(\widetilde{E})} \neq 0$.

Suppose $E \in \mathcal{V}_{n-1}(\mathcal{I})$. Since every direction in $H(E)/E$ corresponds to an integral element E^+ in which E is a codimension one subspace, one might expect the dimension of $H(E)$ to be related to the dimensions of $\mathcal{V}_{n-1}$ at E and $\mathcal{V}_n$ at E^+. This relationship is made precise in the following lemma, which will be important for the proof of Cartan's Test.

Lemma 7.1.10. *Let $E \in \mathcal{V}_{n-1}(\mathcal{I})_p$, $E^+ \in \mathcal{V}_n(\mathcal{I})_p$ and $E \subset E^+ \subset T_p\Sigma$. Then*

$$\begin{aligned} (7.3) \quad & \operatorname{codim}_{E^+}(\mathcal{V}_n(\mathcal{I}), G_n(T\Sigma)) \\ & \qquad \geq \operatorname{codim}_E(\mathcal{V}_{n-1}(\mathcal{I}), G_{n-1}(T\Sigma)) + \operatorname{codim}(H(E), T_p\Sigma). \end{aligned}$$

Proof. Let $s = \dim\Sigma - n$, and take coordinates $x^1, \dots, x^n$, $y^1 \dots y^s$ on Σ, centered at p, such that $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n}$ span E^+, $dx^n|_E = 0$, and $H(E)$ is annihilated by the forms dy^α, where $1 \leq \alpha \leq \operatorname{codim} H(E)$. Then there are linearly independent n-forms Φ^α in $\mathcal{I}$ such that

$$dy^\alpha(v) = \Phi^\alpha(v, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}) \tag{7.4}$$

for all $v \in T_p\Sigma$.

For $\widetilde{E}^+ \in G_n(T\Sigma)$ near E^+, define the functions p^a_i by requiring that the vectors

$$X_i = \frac{\partial}{\partial x^i} + p^a_i \frac{\partial}{\partial y^a}$$

be a basis for $\widetilde{E}^+$. (Here, $1 \leq i \leq n$ and $1 \leq a \leq s$.) Similarly, for $\widetilde{E} \in G_{n-1}(T\Sigma)$ near E, define the functions q^a_j and u_j on $G_{n-1}(T\Sigma)$ by requiring that

$$Z_j = \frac{\partial}{\partial x^j} + u_j \frac{\partial}{\partial x^n} + q^a_j \frac{\partial}{\partial y^a}, \qquad 1 \leq j \leq n-1,$$

span $\widetilde{E}$. The p^a_i are, in fact, part of the local coordinate system on $G_n(T\Sigma)$, in which E^+ is the origin, that we defined before. The q^a_j and u_j also complete a local coordinate system on $G_{n-1}(T\Sigma)$ near E, in which E is the origin.

By the definition of codimension, there are forms $\phi^\nu \in \mathcal{I}^{n-1}$, where $1 \leq \nu \leq \operatorname{codim}_E \mathcal{V}_{n-1}$, such that the functions

$$F^\nu = \phi^\nu(Z_1, \dots, Z_{n-1})$$

have linearly independent differentials at E. Now let

$$G^\nu = \phi^\nu \wedge dx^n(X_1, \dots, X_n),$$
$$H^\alpha = \Phi^\alpha(X_1, \dots, X_n).$$

Certainly, $\mathcal{V}_n(\mathcal{I})$ must lie inside the common zero locus of these $\operatorname{codim}_E \mathcal{V}_{n-1} + \operatorname{codim} H(E)$ functions near E^+. We will show that they have linearly independent differentials at E^+.

Note that F^ν is a polynomial in the q^a_j and u^j, with coefficients depending on x^i and y^a, and G^ν is a polynomial in the p^a_i. In fact $G^\nu(p^a_i) = F^\nu(p^a_j, 0)$, i.e., G^ν is obtained from F^ν by setting $q^a_j = p^a_j$ and $u_j = 0$. (In particular, G^ν does not involve the p^a_n.) Furthermore, since any other codimension one subspace of E^+ is also in $\mathcal{V}_{n-1}(\mathcal{I})$, $dF^\nu|_E\,(\partial/\partial u^j) = 0$. It follows that the G^ν have $\operatorname{codim}_E \mathcal{V}_{n-1}(\mathcal{I})$ linearly independent differentials at E^+.

Let $\Psi^\alpha = \Phi^\alpha - dy^\alpha \wedge dx^1 \wedge \cdots \wedge dx^{n-1}$. Then (7.4) shows that Ψ^α is a sum of terms that either vanish at p, or are wedge products of degree at least two in $(dx^n, dy^1, \dots, dy^s)$. Thus, $\Psi^\alpha(X_1, \dots, X_n)$ will be a polynomial in the p^a_i consisting of terms that either vanish at p, or are of degree two in the p^a_i, or, when they are of degree one, do not involve the p^a_n. It follows that

$$dH^\alpha|_{E^+} \equiv dp^\alpha_n \mod \{dp^a_j, j < n\}.$$

Thus, G^ν and H^α have linearly independent differentials at E^+. $\square$

Kähler-regularity. It is clear from Definition 7.1.6 that the coefficients of the equations defining $H(E)$ depend continuously on E. Thus, the codimension of $H(E)$ is a lower semicontinuous function of E—that is, on a small enough neighborhood of E, the codimension of the polar space can only increase. We will be interested in the case when it is locally constant, so we make the following definition:

Definition 7.1.11. A Kähler-ordinary integral element E is *Kähler-regular* if $\operatorname{codim} H(\widetilde{E}) = \operatorname{codim} H(E)$ for all $\widetilde{E}$ in a neighborhood of E in $\mathcal{V}_n(\mathcal{I})$.

Remark 7.1.12. By convention, the polar space of $0 \in T_p\Sigma$ is the subspace annihilated by the 1-forms of $\mathcal{I}$ at p. Hence the zero subspace in $T_p\Sigma$ is Kähler-regular if $\dim(\mathcal{I}^1)$ is constant on a neighborhood of p.

We would like to find conditions under which the ability to solve the infinitesimal Cauchy problem (i.e., extending E to E^+) implies the ability

to solve the actual Cauchy problem. We will see that Kähler-regularity provides such a condition. The following example shows that if there are no Kähler-regular integral elements that can be extended, then the Cauchy problem may not be solvable.

Example (7.1.8, continued). On any integral 2-plane E^+ of $\mathcal{I}$, two of the forms ω^1, ω^2, π must be independent, and yet $\omega^1 \wedge \pi$ and $\omega^2 \wedge \pi$ must be zero. It follows that $\pi|_{E^+} = 0$, and $E^+ = \{\theta_1, \theta_2, \pi\}^\perp$.

If a 1-dimensional integral element E has $\pi|_E \neq 0$, then E cannot be extended to an integral 2-plane, and $H(E) = E$. (Since $\pi|_E \neq 0$ is an open condition, such E's are Kähler-regular.) If $\pi|_E = 0$, then E is contained in a unique integral 2-plane E^+. Thus, the only integral 1-planes that can be extended are not Kähler-regular, and in fact we now show that the extensions E^+ in general are not tangent to integral surfaces.

Suppose we are trying to construct two-dimensional integral manifolds of $\mathcal{I}$. The structure equations (7.2) imply that the 1-form π vanishes on all such surfaces and therefore must be added to the ideal. (The presence of "extra equations" like $\pi = 0$ is typical when Kähler-regular elements are not available, or cannot be extended.) In fact, when this is done, all the remaining integral 1-planes are Kähler-regular, and we are looking for integral surfaces of the Pfaffian system $\mathcal{J} = \{\theta_1, \theta_2, \pi\}_{\mathsf{diff}}$; these will exist only where $\mathcal{J}$ is Frobenius.

Lemma 7.1.10 gives an upper bound for the dimension of $\mathcal{V}_n(\mathcal{I})$ at E^+ in terms of the polar space $H(E)$. The following lemma shows that this upper bound is achieved when E is Kähler-regular.

Lemma 7.1.13. *Let E, E^+ be as in Lemma 7.1.10. Assume E is Kähler-regular, and let $\phi^\nu \in \mathcal{I}^{n-1}$, $1 \leq \nu \leq \operatorname{codim}_E \mathcal{V}_{n-1}(\mathcal{I})$, be such that the corresponding functions F^ν on $G_{n-1}(T\Sigma)$ have linearly independent differentials at E. Let Φ^α be a collection of linearly independent n-forms in $\mathcal{I}$ such that*

$$H(E) = \{v \in T_p\Sigma | (v \lrcorner \Phi^\alpha)|_E = 0, \forall \alpha\}.$$

Then there are a 1-form θ and a neighborhood U^+ of E^+ in $G_n(T\Sigma)$, such that for every $\widetilde{E}^+ \in U^+$, $\widetilde{E}^+$ is a Kähler-ordinary integral n-plane if and only if $\Phi^\alpha|_{\widetilde{E}^+} = 0$ and $\phi^\nu \wedge \theta|_{\widetilde{E}^+} = 0$ for all α and ν.

Proof. Let $x^1, \ldots, x^n, y^1, \ldots, y^s$, $X_1, \ldots, X_n$, and p_i^a be as in the proof of Lemma 7.1.10. Let $\theta = dx^n$, and let U^+ be the neighborhood of E^+ in $G_n(T\Sigma)$ where the p_i^a are defined.

Since E is Kähler-ordinary, the forms ϕ^ν span $\mathcal{I}^{n-1}$ in the vicinity of p. Since E is Kähler-regular, there is a neighborhood U of E in $G_{n-1}(T\Sigma)$ such that the Φ^α also generate the polar equations for any $\widetilde{E} \in U \cap \mathcal{V}_{n-1}(\mathcal{I})$.

If $\phi^\nu \wedge \theta|_{\widetilde{E}^+} = 0$, then

$$\phi^\nu \wedge \theta(X_1, \dots, X_n) = \phi^\nu(X_1, \dots, X_{n-1}) \tag{7.5}$$

shows that the $(n-1)$-plane $\widetilde{E} \subset \widetilde{E}^+$ spanned by $X_1, \dots, X_{n-1}$ is an integral plane. If $\Phi^\alpha|_{\widetilde{E}^+} = 0$ as well, then $\widetilde{E}^+ \subset H(\widetilde{E})$, and so $\widetilde{E}^+ \in \mathcal{V}_n(\mathcal{I})$.

Now it is clear that $\operatorname{codim}_{E^+} \mathcal{V}_n(\mathcal{I}) \leq \operatorname{codim}_E \mathcal{V}_{n-1}(\mathcal{I}) + \operatorname{codim} H(E)$, and this upper bound holds for all nearby integral n-planes. Then the lower bound (7.3) and the lower semi-continuity of codimension show that E^+ is Kähler-ordinary. □

7.2. Example: Triply orthogonal systems

In this section we discuss an example that uses the Cauchy-Kowalevski Theorem to construct three-dimensional integral manifolds for an EDS. The construction is relatively simple, since the system in the example is generated algebraically by forms of degree three. The example will also serve as an outline for our proof of Cartan-Kähler, since the setup is similar except for details necessary when there are generators of lower degree.

A *triply orthogonal system* in Euclidean space consists of three foliations of $\mathbb{E}^3$ by surfaces that intersect orthogonally at each point. For example, the coordinate surfaces for Cartesian, cylindrical and spherical coordinates all constitute triply orthogonal systems. More generally, a system of *curvilinear coordinates* in Euclidean space consists of three functions u, v, w, with linearly independent differentials, such that the metric $(du)^2 + (dv)^2 + (dw)^2$ is at each point a multiple of the Euclidean metric, and this is the case if and only if the level surfaces of u, v, w form a triply orthogonal system. So, when one asks for metrics on Euclidean space that are conformal to the standard metric, one is essentially asking for a triply orthogonal system of surfaces, from which one can recover u, v, w up to reparametrization. We will show how "large" the family of triply orthogonal systems is.

We associate a frame field (e_1, e_2, e_3) to the foliations of a triply orthogonal system by letting e_i be a smoothly varying unit normal to the ith foliation. This frame field gives rise to a three-dimensional integral manifold of an exterior differential system on the orthonormal frame bundle $\mathcal{F} = \mathcal{F}^{on}_{\mathbb{E}^3}$ (see §2.1) as follows:

Let $D \subset \mathbb{E}^3$ be the domain of the triply orthogonal system, and let $f : D \to \mathcal{F}|_D$ be the frame field. Then f pulls back the canonical forms $(\omega^1, \omega^2, \omega^3)$ on $\mathcal{F}$ to give the coframe field dual to (e_1, e_2, e_3). The Frobenius condition for the distribution defined by $f^*(\omega^i)$ implies that $f^*(d\omega^i \wedge \omega^i) = 0$

for $i = 1, 2, 3$. Since

$$
\begin{aligned}
d\omega^1 \wedge \omega^1 &= -\omega^1_2 \wedge \omega^2 \wedge \omega^1 - \omega^1_3 \wedge \omega^3 \wedge \omega^1,\\
d\omega^2 \wedge \omega^2 &= -\omega^2_3 \wedge \omega^3 \wedge \omega^2 - \omega^2_1 \wedge \omega^1 \wedge \omega^2,\\
d\omega^3 \wedge \omega^3 &= -\omega^3_1 \wedge \omega^1 \wedge \omega^3 - \omega^3_2 \wedge \omega^2 \wedge \omega^3,
\end{aligned}
\tag{7.6}
$$

we see that $f(D)$ is a 3-dimensional integral manifold of the exterior differential system $\mathcal{I}$ generated by $\Omega^1 = \omega^2_3 \wedge \omega^2 \wedge \omega^3$, $\Omega^2 = \omega^3_1 \wedge \omega^3 \wedge \omega^1$, and $\Omega^3 = \omega^1_2 \wedge \omega^1 \wedge \omega^2$ (see Exercises 7.2.1). Conversely, a 3-dimensional integral manifold of $\mathcal{I}$ that satisfies the independence condition $\omega^1 \wedge \omega^2 \wedge \omega^3 \neq 0$ defines the field of normals for a triply orthogonal system on an open set $U \subset \mathbb{E}^3$.

Let $\pi : \mathcal{F} \to \mathbb{E}^3$ be the bundle projection. Since $\mathcal{I}$ contains no forms of degree two or lower, all 2-planes are Kähler-ordinary integral elements. We will restrict our attention to the open subset of $G(2, T_p\mathcal{F})$ consisting of 2-planes E such that $\pi_*|_E$ is injective, since only they are contained in 3-planes that satisfy the independence condition. In fact, each such E is contained in a unique integral 3-plane. For, $\pi_*|_E$ being injective implies that none of the 2-forms $\omega^1 \wedge \omega^2$, $\omega^2 \wedge \omega^3$, and $\omega^3 \wedge \omega^1$ restrict to E to be zero. Then, one sees from the formulae for the Ω^i that the polar equations for E are rank 3, and $\dim H(E) = 3$. Since our assumption about E is an open condition, all such E are Kähler-regular.

We will illustrate below how a frame field for a triply orthogonal system may be extended from a surface to an open domain in $\mathbb{E}^3$ using the Cauchy-Kowalevski Theorem. Since this is a theorem in the real-analytic category, it is necessary to check that the generators of $\mathcal{I}$ are analytic. First of all, the frame bundle is diffeomorphic to the (matrix) Lie group of rigid motions, which endows $\mathcal{F}$ with analytic coordinates. Then, since via this diffeomorphism the ω's correspond to entries in the Maurer-Cartan form $g^{-1}dg$, the ω's are analytic differential forms—that is, combinations of differentials of analytic functions with analytic functions as coefficients. Hence we have an analytic EDS.

Assume $P^2 \subset \mathcal{F}$ is an analytic Kähler-regular integral surface of $\mathcal{I}$ (i.e., all its tangent planes are Kähler-regular), and fix a point $p \in P$. There exist analytic coordinates $x^0, x^1, x^2, y^1, y^2, y^3$ on a neighborhood U of p in $\mathcal{F}$, such that $P \cap U$ is defined by $x^0 = y^1 = y^2 = y^3 = 0$ and $H(T_pP)$ is spanned by $\dfrac{\partial}{\partial x^0}, \dfrac{\partial}{\partial x_1}$, and $\dfrac{\partial}{\partial x_2}$. We will construct an analytic 3-dimensional integral manifold N, containing $P \cap U$, given by $y^\alpha = F^\alpha(x^0, x^1, x^2)$ for some functions F^α to be determined. If N is of this form, the vectors

$\dfrac{\partial}{\partial x^0} + \dfrac{\partial F^\alpha}{\partial x^0}\dfrac{\partial}{\partial y^\alpha}$ and $\dfrac{\partial}{\partial x^i} + \dfrac{\partial F^\alpha}{\partial x^i}\dfrac{\partial}{\partial y^\alpha}$ will span $T_{\pi(p)}N$. (In what follows, Greek indices run from 1 to 3 and Roman indices from 1 to 2.)

We will now show that the F^α satisfy a system of PDE of Cauchy-Kowalevski type. For variables q^β and p^β_j, let

$$\Omega^\alpha\left(\frac{\partial}{\partial x^0} + q^\beta\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^1} + p^\beta_1\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^2} + p^\beta_2\frac{\partial}{\partial y^\beta}\right) = A^\alpha_\beta q^\beta + B^\alpha,$$

where A^α_β and B^α are polynomial in the p^β_j with analytic functions on U as coefficients. Since at p, the equations

$$\Omega^\alpha\left(\frac{\partial}{\partial x^0} + q^\beta\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) = 0$$

coincide with the polar equations of E, they are of rank 3 in the q^β. We may assume that A^α_β is invertible on some (possibly smaller) $U \subset \mathcal{F}$ and for $\|p^\beta_j\|$ sufficiently small. Then

$$(A^{-1})^\gamma_\alpha\Omega^\alpha\left(\frac{\partial}{\partial x^0} + q^\beta\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^1} + p^\beta_1\frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^2} + p^\beta_2\frac{\partial}{\partial y^\beta}\right) = q^\gamma - C^\gamma,$$

where C^γ is an analytic function of the x's, y's and the p^β_j. So, we must have

$$\frac{\partial F^\alpha}{\partial x^0} = C^\alpha\left(x^0, x^1, x^2, F^1, F^2, F^3; \frac{\partial F^\beta}{\partial x^j}\right)$$

with $F^\alpha(0, x^1, x^2) = 0$. Since this is now a system of PDE in Cauchy-Kowalevski form, we conclude that a unique solution exists, on a possibly smaller U.

Any discussion of triply orthogonal systems must mention the famous theorem of Dupin that the surfaces in the system intersect along lines of curvature. This is easy to prove using our system $\mathcal{I}$, since one has only to check that along a surface to which e_3 is normal, e_1 and e_2 diagonalize the second fundamental form. The theorem of Dupin is most striking when applied to the triply orthogonal system formed by the confocal quadrics

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1, \quad a^2 < b^2 < c^2,$$

as λ varies. One sees that along the ellipsoids formed when $\lambda < a^2$, the lines of curvature are exactly the intersections with the two kinds of hyperboloids formed when $a^2 < \lambda < b^2$ and $b^2 < \lambda < c^2$ ([**146**], §2-11).

Exercises 7.2.1:

1. Use the structure equations (7.6) to show that the ideal generated by the forms $d\omega^i \wedge \omega^i$ is the same as that generated by the Ω^i. ⊚

2. An integral surface $P \subset \mathcal{F}$ corresponds to a surface $S \subset \mathbb{E}^3$ with a frame field (e_1, e_2, e_3) attached at each point. Show that if P is Kähler-regular, then none of the vectors e_i can be tangent to S.
3. Use Dupin's theorem to show that any hyperboloid in $\mathbb{E}^3$ has a one-parameter family of closed lines of curvature. ⊚

7.3. Statement and proof of Cartan-Kähler

It is important to note that the Cartan-Kähler Theorem and Cartan's Test only apply to systems generated by real-analytic differential forms on an analytic manifold Σ. (However, a version of Cartan-Kähler is available for *involutive hyperbolic systems* [**155**].) For the rest of this chapter our remarks are limited to the analytic category.

Theorem 7.3.1 (First version of Cartan-Kähler). *Assume $\mathcal{I}$ is an analytic EDS on Σ and $P^n \subset \Sigma$ is an analytic submanifold whose tangent spaces are Kähler-regular integral elements such that, at each $p \in P$, $H(T_pP)$ has dimension $n+1$. Then, for each $p \in P$, there is an open neighborhood $U \subset \Sigma$ of p and a unique analytic $(n+1)$-dimensional integral manifold $N \subset U$ containing $P \cap U$.*

Proof. As in §7.2, we will reduce our problem to solving a system using the Cauchy-Kowalevski Theorem.

Choose analytic coordinates $x^0, x^1, \dots, x^n$, $y^1, \dots, y^s$, centered at p, such that $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n}$ span T_pP and $\dfrac{\partial}{\partial x^0} \in H(T_pP)$. As in Lemma 7.1.13, let $\{\phi^\nu\}$ be a basis for $\mathcal{I}^n$ near p, and let the $(n+1)$-forms Φ^α, $1 \le \alpha \le s$, generate the polar equations in a neighborhood of T_pP.

If N^{n+1} is an integral manifold containing P, then $T_pN = H(T_pP)$, and the x's serve as local coordinates on N. If N is to be defined by $y^\alpha = F^\alpha(x^0, x^1, \dots, x^n)$, then TN will be spanned by $X_0 = \dfrac{\partial}{\partial x^0} + \dfrac{\partial F^\alpha}{\partial x^0}\dfrac{\partial}{\partial y^\alpha}$ and the vectors $X_i = \dfrac{\partial}{\partial x^i} + \dfrac{\partial F^\alpha}{\partial x^i}\dfrac{\partial}{\partial y^\alpha}$. Now define functions $A^\alpha_\beta(x^i, y^\gamma; p^\gamma_j)$ and $B^\alpha(x^i, y^\gamma; p^\gamma_j)$ by

$$\Phi^\alpha\left(\frac{\partial}{\partial x^0} + q^\beta \frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^1} + p^\beta_1 \frac{\partial}{\partial y^\beta}, \dots, \frac{\partial}{\partial x^n} + p^\beta_n \frac{\partial}{\partial y^\beta}\right) = A^\alpha_\beta q^\beta + B^\alpha.$$

(We let $1 \le \alpha, \beta, \gamma \le s$.) As in §7.2, the A^α_β are the entries of an invertible matrix when evaluated at the origin; so, on a possibly smaller open set (which we continue to denote by U) and for sufficiently small $\|p^\beta_j\|$, we can

define forms $\widetilde{\Phi}^\alpha = (A^{-1})^\alpha_\beta \Phi^\beta$. Let $C^\alpha(x^i, y^\beta; p^\beta_j)$ be defined by

$$\widetilde{\Phi}^\alpha \left(\frac{\partial}{\partial x^0} + q^\beta \frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial x^1} + p^\beta_1 \frac{\partial}{\partial y^\beta}, \ldots, \frac{\partial}{\partial x^n} + p^\beta_n \frac{\partial}{\partial y^\beta} \right) = q^\alpha - C^\alpha.$$

Then we construct N by using Cauchy-Kowalevski to solve

$$\begin{cases} \dfrac{\partial F^\alpha}{\partial x^0} = C^\alpha \left(x^0, x^1, \ldots, x^n, F^1, \ldots, F^s; \dfrac{\partial F^\beta}{\partial x^j} \right), \\ F^\alpha(0, x^1, \ldots, x^n) = 0. \end{cases}$$

It remains to be shown that the forms $\phi^\nu \wedge dx^0$, which by 7.1.13 span the rest of the $(n+1)$-forms in $\mathcal{I}$, pull back to be zero on N. By (7.5), this is equivalent to showing that $\phi^\nu(X_1, \ldots, X_n) = 0$. On N,

$$\frac{\partial}{\partial x^0} \phi^\nu(X_1, \ldots, X_n) = d\phi^\nu(X_0, \ldots, X_n) - \sum_i \frac{\partial}{\partial x^i} \left(dx^i \wedge \phi^\nu(X_0, \ldots, X_n) \right).$$

However, since $d\phi^\nu \equiv 0$ and $\phi^\nu \wedge dx^i \equiv 0$ modulo $\{\Phi^\alpha, \phi^\nu \wedge dx^0\}$, the right-hand side of this PDE is expressible in terms of analytic functions of the $\phi^\nu(X_1, \ldots, X_n)$ and their x^i-derivatives. Since $\phi^\nu(X_1, \ldots, X_n) = 0$ along $x^0 = 0$, by the Cauchy-Kowalevski Theorem the *unique* solution to this system of partial differential equations is $\phi^\nu(X_1, \ldots, X_n) \equiv 0$ for all ν. □

Theorem 7.3.2 (Second version of Cartan-Kähler). *Assume Σ, $\mathcal{I}$ and P^n are as in the first version, but $H(T_pP)$ has dimension $n + r + 1$ for each $p \in P$. Assume that $R \subset \Sigma$ is an analytic submanifold of codimension r, such that $P \subset R$ and T_pR intersects transversely with $H(T_pP)$. Then for each $p \in P$ there is a neighborhood $U \subset R$ of p and a unique analytic $(n+1)$-dimensional integral manifold $N \subset U$ containing $P \cap U$.*

Proof. The transversality condition implies that the intersections have dimension $n + 1$. Then apply the previous version with Σ replaced by R. □

The Cartan-Kähler Theorem can be applied inductively to obtain an integral manifold with a given integral n-plane E as its tangent space. To do this, one needs a flag $0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$ of integral elements inside E, such that the second version of the theorem can be applied at each step. As is evident in the statement of the last theorem, one needs to choose a "restraining manifold" R at each step, in order to make the Cauchy problem determined. The following result gives an idea of how this is done.

Theorem 7.3.3 (Third version of Cartan-Kähler). *Let E_k, $0 \le k \le n$, be a flag of integral elements at p for an analytic EDS, with $\dim E_k = k$, and such that E_k is Kähler-regular for $0 \le k \le n-1$. Then there exists a smooth n-dimensional integral manifold N whose tangent space at p is E_n. Furthermore, let $c_k = \operatorname{codim} H(E_k)$ for $0 \le k \le n$, and let $x^1, \ldots, x^n$*

and $y^1, \dots, y^s$ be local coordinates on Σ, centered at p, chosen so that E_k is spanned by $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^k}$ and $H(E_k)$ is annihilated by $dy^1, \dots, dy^{c_k}$. If N is defined in these coordinates by the equations $y^a = F^a(x^1, \dots, x^n)$, $1 \le a \le s$, then N is uniquely determined by the data

$$\begin{aligned} f^a(x^1) &= F^a(x^1, 0, \dots, 0) && \text{for } c_0 < a \le c_1, \\ f^a(x^1, x^2) &= F^a(x^1, x^2, 0, \dots, 0) && \text{for } c_1 < a \le c_2, \\ &\dots \end{aligned}$$

Conversely, there exists an N corresponding to the data if the functions f^a are sufficiently small.

Proof. Let R_1 be the manifold given by setting $x^2 = \cdots = x^n = 0$ and $y^a = F^a(x^1, 0, 0, \dots)$ for $a > c_0$. By Theorem 7.3.2, using $R = R_1$, there exists a unique 1-dimensional integral manifold inside R_1, containing p and tangent to E_1 at p. This must be the intersection of N with the set where $x^2 = \cdots = x^n = 0$. Proceeding in this way, letting R_k be defined by $x^{k+1} = \cdots = x^n = 0$ and $y^a = f^a(x^1, \dots, x^k, 0, 0, \dots)$ for $a > c_{k-1}$, we can reconstruct N uniquely from the data. (We need to assume the f's are small enough that the transversality condition is satisfied.) $\square$

7.4. Cartan's Test

In order to obtain a flag of Kähler-regular integral elements, it seems that it would be necessary to compute not only the dimensions of the polar spaces $H(E_k)$, but also the dimensions of the polar spaces of all nearby integral elements! This is avoided by the following test, which generalizes (4.18) and Definition 4.5.4:

Theorem 7.4.1 (Cartan's Test for Involutivity). *Let E_k, $0 \le k \le n$, be a flag of integral elements for $\mathcal{I}$ at p, and let $c_k = \operatorname{codim} H(E_k)$ for $0 \le k \le n-1$. Then*

$$\operatorname{codim}_{E_n} \mathcal{V}_n(\mathcal{I}) \ge c_0 + c_1 + \cdots + c_{n-1}. \tag{7.7}$$

Moreover, $\mathcal{V}_n(\mathcal{I})$ is smooth of codimension exactly $c_0 + c_1 + \cdots + c_{n-1}$ at E_n if and only if the E_k are all Kähler-regular for $0 \le k \le n-1$.

Definition 7.4.2. If there exists a flag for E with equality holding in (7.7), then we say E is an *ordinary* integral element. If there is a neighborhood of E of ordinary integral elements, we say that $\mathcal{I}$ is *involutive at E*.

Remark 7.4.3. If an integral element E is equipped with a flag for which the characters fail Cartan's Test, one must make sure the flag was chosen

generically. If the characters for a generic flag still fail the test, it is necessary to prolong the system (see §5.5).

Proof of 7.4.1. The inequality follows by successive application of (7.3). Furthermore, if all the E_k are Kähler-regular, successive applications of Lemma 7.1.13 show that $\mathcal{V}_n(\mathcal{I})$ is smooth at E_n, with the required codimension.

Now suppose $\mathcal{V}_n(\mathcal{I})$ is smooth at E_n, with the above codimension. We first show that E_{n-1} is Kähler-ordinary. One might suspect this is the case because $\mathcal{V}_n(\mathcal{I})$ cannot be so large without $\mathcal{V}_{n-1}(\mathcal{I})$ being large. Consider the following heuristic argument:

By the first part of this theorem, $\mathcal{V}_{n-1}(\mathcal{I})$ has codimension at least $c_0 + \cdots + c_{n-2}$ at E_{n-1}. On the other hand, for every $\widetilde{E}_n \in \mathcal{V}_n(\mathcal{I})$ near E_n, all the hyperplanes inside $\widetilde{E}_n$ are in $\mathcal{V}_{n-1}(\mathcal{I})$. In fact we may parametrize this set of hyperplanes, and get a map from $\mathcal{V}_n(\mathcal{I}) \times \mathbb{R}^{n-1}$ to a neighborhood of E_{n-1} in $\mathcal{V}_{n-1}(\mathcal{I})$. But for $\widetilde{E}_{n-1} \in \mathcal{V}_{n-1}(\mathcal{I})$ sufficiently close to E_{n-1}, $H(\widetilde{E}_{n-1})$ has codimension at least c_{n-1}. This means the map has fiber of dimension at most $s - c_{n-1}$. (Here we have set $s = \dim \Sigma - n$.) Since the dimension of $G_n(T\Sigma)$ is $n + s + ns$, $\mathcal{V}_n(\mathcal{I})$ has dimension

$$d = n + s + ns - (c_0 + \cdots + c_{n-1}).$$

So, we expect the image of this map to have dimension at least

$$d + n - 1 - (s - c_{n-1} - 1) = n + s + (n-1)(s+1) - (c_0 + \cdots + c_{n-2}),$$

which is exactly the largest dimension it can be. We will now make this argument rigorous.

Let $x^1, \ldots, x^n, y^1, \ldots, y^s$ be local coordinates such that E_k is spanned by $\partial/\partial x^1, \ldots, \partial/\partial x^k$ and $H(E_k)$ is annihilated by $dy^1, \ldots, dy^{c_k}$. As in the proof of Lemma 7.1.10, we define functions p^a_i completing a coordinate system on $U \subset G_n(T\Sigma)$ centered at E_n, and also define X_i, q^a_j, u_j and Z_j as before.

Let $F : U \times \mathbb{R}^{n-1} \to G_{n-1}(T\Sigma)$ take $(\widetilde{E}_n, (v_1, \ldots, v_{n-1}))$ to the subspace of $\widetilde{E}_n$ spanned by the vectors $X_j + v_j X_n$, $1 \leq j \leq n-1$. In coordinates, this map is given by $u_j = v_j$ and $q^a_j = p^a_j + v_j p^a_n$. Differentiating these equations shows that the kernel of F_* at $(E_n, (0, \ldots, 0))$ is spanned by the vectors $\partial/\partial p^a_n$.

Let f denote the restriction of F to $(\mathcal{V}_n(\mathcal{I}) \cap U) \times \mathbb{R}^{n-1}$. We want to show that the rank of f is at least $d - (s - c_{n-1})$ at $(E_n, (0, \ldots, 0))$, or equivalently that the intersection of the kernel of F_* with $T_{E_n}\mathcal{V}_n(\mathcal{I})$ has dimension at most $s - c_{n-1}$. Once this is known, the image of f contains a smooth submanifold through E_{n-1} with maximum codimension, and it

follows that E_{n-1} is Kähler-regular; in fact, the image of f fills out $\mathcal{V}_{n-1}(\mathcal{I})$ near E_{n-1}.

We will need to look at the equations that cut out $\mathcal{V}_n(\mathcal{I})$ near E_n in local coordinates. By the above inequality, $\operatorname{codim}_{E_{n-1}} \mathcal{V}_{n-1}(\mathcal{I}) \geq c_0 + \cdots + c_{n-2}$. So, let ϕ^ν, $1 \leq \nu \leq c_0 + \cdots + c_{n-2}$, and Φ^α, $1 \leq \alpha \leq c_{n-1}$, be defined as in the proof of Lemma 7.1.10. We saw then that the n-forms $\phi^\nu \wedge dx^n$ and Φ^α give rise to functions G^ν and H^α on $G_n(T\Sigma)$ with linearly independent differentials at E_n. Since $\mathcal{V}_n(\mathcal{I})$ is smooth at E_n, and of exactly this codimension, the equations $G^\nu = 0$ and $H^\alpha = 0$ define $\mathcal{V}_n(\mathcal{I})$ near E_n.

As before, G^ν is a polynomial in the p^a_i which does not involve the p^a_n, and

$$dH^\alpha|_{E_n} \equiv dp^\alpha_n \mod \{dp^a_i, i < n\}, \qquad 1 \leq \alpha \leq c_{n-1}. \tag{7.8}$$

Now, if some linear combination $\sum_{a=1}^s t_a \partial/\partial p^a_n$ is tangent to $\mathcal{V}_n(\mathcal{I})$ at E_n, (7.8) shows that $t_a = 0$ for $1 \leq a \leq c_{n-1}$. It follows that the kernel of f_* at E_n is as small as desired, and the rank of f is as large as desired.

Now it is easy to show that E_{n-1} is Kähler-regular. For, suppose a nearby $\widetilde{E}_{n-1} \in \mathcal{V}_{n-1}(\mathcal{I})$ has $\operatorname{codim} H(\widetilde{E}_{n-1}) > \operatorname{codim} H(E_{n-1})$. This $\widetilde{E}_{n-1}$ will be in the image of f, and so is contained in an $\widetilde{E}_n \in \mathcal{V}_n(\mathcal{I})$ near E_n. Now (7.3) implies that the codimension of $\mathcal{V}_n(\mathcal{I})$ at $\widetilde{E}_n$ is larger than it really is, so we reach a contradiction.

To conclude the proof of the theorem, we apply the above argument inductively to show that every E_k is Kähler-regular. □

Characters. In practice, the inequality of Cartan's Test is used in a different form. Assume that E is a Kähler-ordinary integral n-plane with a chosen flag, and set

$$\begin{aligned}
s_0 &= c_0, \\
s_k &= c_k - c_{k-1}, \qquad 1 \leq k \leq n-1, \\
s_n &= \operatorname{codim} E - c_{n-1} = \operatorname{codim} E - (s_0 + s_1 + \cdots + s_{n-1}).
\end{aligned}$$

Then $\dim \mathcal{V}_n(\mathcal{I}) - \dim \Sigma \leq s_1 + 2s_2 + 3s_3 + \cdots + ns_n$, with equality if and only if E is ordinary and a sufficiently generic flag has been chosen.

Once an EDS has passed Cartan's Test, the Cartan-Kähler Theorem 7.3.3 may be used to show that integral manifolds exist. Moreover, 7.3.3 shows that manifolds depend on successive choices of s_0 constants, s_1 functions of one variable, s_2 functions of two variables, etc., just as in Theorem 4.5.7.

Remark 7.4.4 (Linear Pfaffian Systems). When we have a linear Pfaffian system with structure equations

$$d\theta^a \equiv \pi^a_i \wedge \omega^i \mod \theta^1, \ldots, \theta^s,$$

and E an integral element satisfying the independence condition, there will be 1-forms $\tilde{\pi}_i^a = \pi_i^a - p_{ij}^a\omega^j$, with $p_{ij}^a = p_{ji}^a$, such that $\tilde{\pi}_i^a|_E = 0$. (In fact, we can exchange π for $\tilde{\pi}$ in the structure equations.) Choose the flag so that $\omega_i|_{E_k} = 0$ for $i > k$, and let $e_1, \dots, e_n$ be the basis for E dual to the ω^i.

Of course, $H(E_0)$ is just the subspace of $T_p\Sigma$ annihilated by the θ's. Applying 7.1.7.5 shows that $H(E_k)$ is annihilated by the θ's and by the forms $\tilde{\pi}_i^a$ for $i \leq k$. Thus, s_k is the number of forms in the kth column of the tableau that are independent of those in previous columns. So, these characters—with the exception of s_n when there are Cauchy characteristics—agree with those defined in Chapter 4.

Remark 7.4.5 (Cauchy characteristics)**.** Note that Cauchy characteristic vectors (see §6.1) will lie inside $H(E)$ for every E. If the space $A(\mathcal{I})$ of Cauchy characteristics is of dimension k, it adds k to s_n (but not to any of the other characters of the flag, since they are differences of dimensions of polar spaces). There will also be k independent 1-forms that are not involved in the Cartan system of $\mathcal{I}$, so these forms also add nk to the fiber dimension of $\mathcal{V}_n(\mathcal{I})$. Thus, the Cauchy characteristics will contribute an extra nk to each side of the inequality in Cartan's Test. What this means is that, if we choose to ignore Cauchy characteristics (i.e., omitting characteristic directions from $\operatorname{codim} E$ and the fiber dimension of $\mathcal{V}_n$), then the result of Cartan's Test will be the same. (This idea is used in Examples 7.5.3 and 7.5.4 below.) When we ignore characteristics in this fashion, we are in effect working on the manifold obtained by forming the quotient by the characteristics, with the quotient system constructed in Theorem 6.1.20.

7.5. More examples of Cartan's Test

We will begin with some examples that are typical of the applications of exterior differential systems to classical surface theory found in Cartan's treatise [**31**]. As in Chapter 2, we let $\mathcal{F} = \mathcal{F}_{\mathbb{E}^3}^{on}$ denote the orthonormal frame bundle of $\mathbb{E}^3$. Given a surface $M \subset \mathbb{E}^3$, recall that a section $f : M \to \mathcal{F}|_M$ is a first-order adapted framing if T_xM is spanned by (f^*e_1, f^*e_2). (As usual, we suppress s and the base point in the notations). Along such a framing, $\omega^3 = 0$ and $\omega_i^3 = h_{ij}\omega^j$, where h_{ij} is the symmetric matrix of the second fundamental form in the e_1, e_2 basis.

Example 7.5.1 (*Surfaces with one of the principal curvatures constant*)**.** (Cartan [**31**], Ex. III) Along such a surface, away from the umbilic points, one can obtain a smooth Darboux framing so that e_2 points in the principal direction with constant principal curvature k_0. Since e_1 points in the other principal direction, this implies that h_{ij} is diagonal with $h_{22} = k_0$. Thus, the framing $f(M)$ is an integral surface of the forms ω^3, $\omega_2^3 - k_0\omega^2$ and $\omega_1^3 \wedge \omega^1$ on $\mathcal{F}$.

Let $\theta = \omega^3_2 - k_0\omega^2$. Since $d\theta = (\omega^3_1 - k_0\omega^1) \wedge \omega^2_1$ and $d\omega^3 \equiv 0$ modulo θ and $\omega^3_1 \wedge \omega^1$, we will let $\pi = \omega^3_1 - k_0\omega^1$ to simplify calculations. Then $f(M)$ is an integral surface of the differential ideal $\mathcal{I} = \{\omega^3, \theta, \pi \wedge \omega^1, \pi \wedge \omega^1_2\}_{\mathsf{alg}}$, to which we now apply Cartan's Test.

First, we investigate the space of integral 2-planes E satisfying the independence condition. While ω^3 and θ must vanish on E, suppose $\pi = a\omega^1$ and $\omega^1_2 = b\omega^1 + c\omega^2$ on E; then the generator 2-forms imply that $ac = 0$. Thus, E is Kähler-ordinary when exactly one of a and c is zero; then the fiber dimension of $\mathcal{V}_2(\mathcal{I})$ is two.

Case 1: Assume $c \neq 0$ and $a = 0$. Let e_1 and e_2 be vectors in E dual to the restrictions of ω^1 and ω^2 to E. Then

$$e_1 \lrcorner (\pi \wedge \omega^1) = -\pi,$$
$$e_1 \lrcorner (\pi \wedge \omega^1_2) = -b\pi.$$

This shows that $H(\{e_1\}) = \{\omega^3, \pi, \theta\}^\perp$. Since $s_0 = 2$, $s_1 = 1$, and $s_2 = 1$, $s_1 + 2s_2$ exceeds the fiber dimension and Cartan's Test fails to show that E is ordinary. Since

$$e_2 \lrcorner (\pi \wedge \omega^1) = 0,$$
$$e_2 \lrcorner (\pi \wedge \omega^1_2) = a\pi,$$

we get the same result for the flag $0 \subset \{e_2\} \subset E$, and in fact for any flag inside E, so E is not ordinary.

If we consider the subvariety $X \subset \mathcal{V}_2(\mathcal{I})$ of integral elements with a=0, the system is not involutive at any $E \in X$. However, if we prolong at points of X, we get a Frobenius system whose solutions are spheres of radius $\frac{1}{k_0}$.

Case 2: Assume $a \neq 0$ and $c = 0$. Let e_1 and e_2 be vectors in E as before. Then

$$e_1 \lrcorner (\pi \wedge \omega^1) = a\omega^1 - \pi$$
$$e_1 \lrcorner (\pi \wedge \omega^1_2) = a\omega^1_2 - b\pi.$$

So, for the flag $0 \subset \{e_1\} \subset E$ we have $s_0 = 2$, $s_1 = 2$, $s_2 = 0$; then $s_1 + 2s_2 = 2$, and this E is ordinary.

On the other hand, suppose we had used the flag $0 \subset \{e_2\} \subset E$. Since

$$e_2 \lrcorner (\pi \wedge \omega^1) = 0,$$
$$e_2 \lrcorner (\pi \wedge \omega^1_2) = 0,$$

we would have had $s_0 = 2$, $s_1 = 0$ and $s_2 = 2$.

We conclude that there exist many of these surfaces: in Case 2, the integral surfaces through a given point in $\mathcal{F}$ depend on $s_1 = 2$ functions of one variable. Once we know this, we ought to find a geometric interpretation for the two arbitrary functions.

Suppose we have one of these surfaces, with a framing adapted as above. Let s be an arclength parameter along a line of curvature in the surface in the e_1 direction. If x represents the coordinate vector for the point on the surface, then $dx/ds = e_1$ along this curve. By differentiating the frame vectors modulo ω^2 and $\mathcal{I}$, we obtain the following system of ordinary differential equations for the frame components along this curve:

$$\frac{dx}{ds} = e_1, \quad \frac{de_1}{ds} = -be_2 + (a + k_0)e_3, \quad \frac{de_2}{ds} = be_1, \quad \frac{de_3}{ds} = -(a + k_0)e_1.$$

Conversely, if we specify functions $a(s) \neq 0$ and $b(s)$, then we can integrate the above system to obtain a Kähler-regular integral curve of $\mathcal{I}$ that has a unique extension to an integral surface.

Exercise 7.5.2: Show that the curvature and torsion of this curve are determined by $\kappa = |q|$ and $\tau = \mathsf{Im}(q'/q)$, where $q(s) = (a + k_0) - ib$. (Note that the real and imaginary parts of q are the curvatures for a "natural" or "relatively parallel" frame [**12**] along the curve.) ◎

Example 7.5.3 (*Linear Weingarten surfaces*)**.** A surface is a *Weingarten surface* if its Gauss curvature and mean curvature satisfy a given equation $f(H, K) = 0$. Recall from §6.4 that when this equation is linear in H and K, the corresponding differential system is locally equivalent to a Monge-Ampère equation. Examples include minimal surfaces ($H = 0$) and pseudospherical surfaces ($K = -1$), both of which are discussed in detail in §6.4.

If $f(H, K) = AK + 2BH + C$ for constants A, B, C, then first-order adapted framings along the surface are integrals of the 2-form

$$\Theta = A\omega^3_1 \wedge \omega^3_2 + B(\omega^3_1 \wedge \omega^2 - \omega^3_2 \wedge \omega^1) + C\omega^1 \wedge \omega^2.$$

Let $\mathcal{I} = \{\omega^3, d\omega^3, \Theta\}_{\mathsf{alg}}$; note that this is not a linear Pfaffian system. However, its prolongation was shown to be involutive in Example 5.8.2.

We now apply Cartan's Test to $\mathcal{I}$. An integral 2-plane E satisfying the independence condition $\omega^1 \wedge \omega^2|_E \neq 0$ will be determined by coefficients h_{ij} such that $\omega^3_i|_E = h_{ij}\omega^j$, and by the values of $\omega^2_1|_E$. (However, since ω^2_1 is dual to a Cauchy characteristic for the system, according to Remark 7.4.5 we may ignore it in computing Cartan characters and the dimension of the space of integral elements. This means we are essentially working on the quotient five-manifold) If h is the symmetric matrix with components h_{ij}, we must have

$$A \det h + B \operatorname{tr} h + C = 0.$$

Thus, $\mathcal{V}_2(\mathcal{I})$ is smooth, of fiber dimension two, except where $Ah + BI = 0$. (This can only happen if $AC - B^2 = 0$; at any rate, we exclude points where $\mathcal{V}_2(\mathcal{I})$ is not smooth.)

Suppose $e \in E$ satisfies $e \lrcorner\, \omega^i = x^i$. Let e span a line $E_1 \subset E$; we will calculate the characters for this flag. Of course, ω^3 must vanish on $H(E_1)$, but so must

$$\left.\begin{aligned} e \lrcorner\, d\omega^3 &\equiv x^1\omega_1^3 + x^2\omega_2^3 \\ e \lrcorner\, \Theta &\equiv -(Ah_{2j}x^j + Bx^2)\omega_1^3 + (Ah_{1j}x^j + Bx^1)\omega_2^3 \end{aligned}\right\} \mod \omega^1, \omega^2.$$

Thus $s_1 = 2$, unless the determinant $Ah_{ij}x^ix^j + B((x^1)^2 + (x^2)^2)$ vanishes. Because we assume that $Ah + BI \neq 0$, we can always choose $e \subset E$ so that this doesn't happen.

So, $H(E_1) = E$, $s_2 = 0$ and $s_1 + 2s_2 = 2$. We conclude that the system for linear Weingarten surfaces is involutive wherever the space of integral 2-planes is smooth, and that solutions depend on two functions of one variable. For example, a minimal surface could be determined (up to rigid motion) by specifying the curvature and torsion of a curve on the surface.

Example 7.5.4 (*Hypersurfaces of constant scalar curvature in S^4*)**.** To set up this example, we will use a trick that extracts the components of the Ricci tensor from the curvature forms of a Riemannian manifold.

Given orthonormal vectors $\omega^1, \ldots, \omega^n$, we let $\omega_{(i_i \cdots i_k)}$ denote the wedge product of $n-k$ of the ω's such that

$$\omega_{(i_i \cdots i_k)} \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} = \omega^1 \wedge \cdots \wedge \omega^n.$$

For example, $\omega_{(2)} = \omega^3 \wedge \omega^1$ when $n = 3$ and $\omega_{(41)} = -\omega^2 \wedge \omega^3$ when $n = 4$. It is easy to verify the identities

$$\begin{aligned} \omega_{(j)} \wedge \omega^i &= \delta_j^i\, \omega^1 \wedge \cdots \wedge \omega^n, \\ \omega_{(jk)} \wedge \omega^i &= \delta_j^i \omega_{(k)} - \delta_k^i \omega_{(j)}, \\ \omega_{(ijk)} \wedge \omega^p &= \delta_i^p \omega_{(jk)} - \delta_j^p \omega_{(ik)} + \delta_k^p \omega_{(ij)}. \end{aligned}$$

Recall from §2.6 that the components of the curvature tensor are determined by the curvature 2-forms:

$$\Theta_j^i = \tfrac{1}{2} R^i{}_{jkl}\, \omega^k \wedge \omega^l. \tag{7.9}$$

Now, if we set $\Phi^{ij} = \Theta_k^i g^{kj}$, then from (7.9) and the above identities we obtain

$$\omega_{(pij)} \wedge \Phi^{ij} = 2R_j^j \omega_{(p)} - 4R_p^j \omega_{(j)},$$

where R_{ij} are the components of the Ricci tensor, and $R_j^i = g^{ik}R_{kj}$. Finally, this equation yields an expression for the scalar curvature R if we wedge with ω^p and sum on the index p:

$$\omega_{(pij)} \wedge \Phi^{ij} \wedge \omega^p = 2(n-2)R\, \omega^1 \wedge \ldots \wedge \omega^n.$$

In particular, on a three-manifold this says $\Phi^{12} \wedge \omega^3 + \Phi^{23} \wedge \omega^1 + \Phi^{31} \wedge \omega^2 = R\,\omega^1 \wedge \omega^2 \wedge \omega^3$. We will use this identity to set up an EDS whose

3-dimensional integral manifolds correspond to hypersurfaces of constant scalar curvature R_0 in S^4.

We will work on $\mathcal{F} = \mathcal{F}^{on}_{S^4} \simeq O(5)$. Let $(e_0, e_1, e_2, e_3, e_4)$ represent a frame with basepoint e_0. Recall that $de_\alpha = e_\beta \omega^\beta_\alpha$, $0 \leq \alpha, \beta \leq 4$, $\omega^\alpha_\beta = -\omega^\beta_\alpha$, and $d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta$ on $\mathcal{F}$. The forms $\omega^i = \omega^i_0$ furnish a basis of the semi-basic forms on $\mathcal{F}$, and ω^i_j, $1 \leq i, j \leq 4$, are the Levi-Civita connection forms for the standard metric on S^4.

Along a hypersurface in S^4 we can choose a framing so that e_4 is always normal to the hypersurface; this will give a section of $\mathcal{F}$ which is a three-dimensional integral manifold of the 1-form ω^4, and to which $\Omega = \omega^1 \wedge \omega^2 \wedge \omega^3$ restricts to be nonzero. (This will be our independence condition.) From now on, let i, j, k be indices that run from 1 to 3. Since $d\omega^i \equiv -\omega^i_j \wedge \omega^j \mod \omega^4$, and the ω^i_j are skew-symmetric, the connection forms ϕ^i_j for the Levi-Civita connection on the hypersurface coincide with ω^i_j. The curvature forms are given by

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \omega^4_i \wedge \omega^4_j + \omega^i \wedge \omega^j.$$

Thus, framings along hypersurfaces of constant scalar curvature R_0 will also be integral manifolds of the 3-form

(7.10)
$$\Upsilon = \omega^4_1 \wedge \omega^4_2 \wedge \omega^3 + \omega^4_2 \wedge \omega^4_3 \wedge \omega^1 + \omega^4_3 \wedge \omega^4_1 \wedge \omega^2 - (R_0 - 3)\omega^1 \wedge \omega^2 \wedge \omega^3.$$

Since $d\omega^4 \equiv -\omega^4_i \wedge \omega^i \mod \omega^4$, we will let $\mathcal{I} = \{\omega^4, \omega^4_i \wedge \omega^i, \Upsilon\}_{\mathsf{alg}}$. (Although we don't care about forms in $\mathcal{I}$ of degree higher than three, one can check that $d\Upsilon \equiv 0 \mod \mathcal{I}$.)

We must calculate the fiber dimension of $\mathcal{V}_3(\mathcal{I}, \Omega)$ before applying Cartan's Test. As explained in Remark 7.4.5, we may ignore the Cauchy characteristics, which are dual to the connection forms ω^i_j. An integral 3-plane E will be determined by $\omega^4_i|_E = h_{ij}\omega^j$, where the h_{ij} are the entries of a symmetric 3×3 matrix h. Substituting this into (7.10) gives $\Upsilon = (h_{ii}h_{jj} - h_{ij}h_{ji} - (R_0 - 3))\, \omega^1 \wedge \omega^2 \wedge \omega^3$. So, $\mathcal{V}_3(\mathcal{I}, \Omega)$ is defined by

$$(\mathrm{tr}\, h)^2 - \mathrm{tr}(h^2) = R_0 - 3. \tag{7.11}$$

Differentiation shows that $\mathcal{V}_3(\mathcal{I}, \Omega)$ is a smooth submanifold at points where the entries of h are not all zero. Its fiber dimension is five.

Let E be a smooth point of $\mathcal{V}_3(\mathcal{I}, \Omega)$. Let v_1, v_2, v_3 be the basis of E dual to the restrictions of $\omega^1, \omega^2, \omega^3$ to E, let $E_i \subset E$ be spanned by $v_1, \ldots, v_i$, and of course let E_0 be the zero subspace. Recall that $H(E_0)$ is just the kernel of the 1-forms in $\mathcal{I}$; $H(E_1)$ is cut out by these 1-forms and by those obtained by feeding the vector v_1 into generator 2-forms of $\mathcal{I}$ (cf. 7.1.7.5).

Since

$$v_1 \lrcorner (\omega_i^4 \wedge \omega^i) \equiv -\omega_1^4 \mod \omega^1, \omega^2, \omega^3,$$

we get one additional 1-form, and $s_1 = \operatorname{codim} H(E_1) - \operatorname{codim} H(E_0) = 1$.

Similarly, $v_2 \lrcorner (\omega_i^4 \wedge \omega^i) \equiv -\omega_2^4 \mod \omega^i$, so $s_2 \geq 1$. But

$$(e_1, e_2) \lrcorner \Upsilon \equiv (h_{11} + h_{22})\omega_3^4 \mod \omega^1, \omega^2, \omega^3, \omega_1^4, \omega_2^4,$$

so if $h_{11} + h_{22} \neq 0$, we get $s_2 = 2$ for this flag. Following 7.4.5, we ignore Cauchy characteristics and take $s_3 = 0$. Then $s_1 + 2s_2 + 3s_3 = 5$, and by the Cartan-Kähler Theorem we can construct an integral manifold through E.

We now address the case when $h_{11} + h_{22} = 0$. If we can obtain an integral manifold that has second fundamental form matrix h with respect to a frame $f \in \mathcal{F}$, then by rotation we can obtain an integral manifold through any other point in the Cauchy characteristic leaf that contains f, and the second fundamental form matrix there will be ghg^{-1}, $g \in SO(3)$. So, it is enough to establish existence of integral manifolds assuming that h is diagonal. In that case, if we cannot arrange that $h_{11} + h_{22} \neq 0$ by cyclically permuting indices, then all three of $h_{11} + h_{22}$, $h_{11} + h_{33}$ and $h_{22} + h_{33}$ must be zero, and this implies that $h = 0$.

We have now proved

Theorem. *Given $p \in S^4$, $E \in G(3, T_pS^4)$, and a nonzero symmetric 3×3 matrix h that satisfies (7.11), there exists a three-dimensional submanifold $N \subset S^4$ with constant scalar curvature R_0, tangent to E, with second fundamental form h at that point.*

We see by Theorem 7.3.3 that the construction of N involves choosing $s_1 = 1$ function of one variable, and $s_2 = 2$ functions of two variables.

Example 7.5.5 (Lagrangian and special Lagrangian submanifolds)**.** Let ω be the standard symplectic form on $\mathbb{R}^{2n}$:

$$\omega = dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n.$$

An n-dimensional submanifold is *Lagrangian* if it is an integral manifold of $\mathcal{I} = \{\omega\}_{\mathsf{alg}}$.

Given $E \in \mathcal{V}_n(\mathcal{I})$, we can make a linear change of coordinates (while keeping the form of ω) so that E is annihilated by $dy^1, \ldots, dy^n$. Any nearby integral n-planes are given by $dy^j = \sum_k s^{jk} dx^j$ for $s^{jk} = s^{kj}$. Therefore, any such E is Kähler-ordinary and the fiber dimension of $\mathcal{V}_n$ is $\binom{n+1}{2}$.

Let $e_1, \ldots, e_n \in E$ be dual to $dx^1, \ldots, dx^n$. Then we find that $s_1 = 1$, $s_2 = 1$, ..., $s_n = 1$. Since $s_1 + 2s_2 + \ldots + ns_n = \binom{n+1}{2}$, we have involutivity, and integral manifolds depend on 1 function of n variables. (In fact, they

can be explicitly constructed by setting $y^j = \partial f/\partial x^j$ for f an arbitrary function of $x^1, \dots, x^n$.)

The preceding computation is a warm-up for applying Cartan's Test to an EDS for special Lagrangian submanifolds.

Recall from §5.12 that these are submanifolds whose tangent spaces belong to the face of the calibration

$$\alpha = \mathsf{Re}\left(dz^1 \wedge \cdots \wedge dz^n\right),$$

where $z^j = dx^j + idy^j$, and the face is defined to be the set of unit-volume n-planes on which the calibration has value one.

In general, any calibration α has a complementary form α_c such that

$$|\alpha(E)|^2 + |\alpha_c(E)|^2 = 1$$

for all unit volume planes E (see [**74**], Theorem 7.104). Thus, $E \in \text{Face}(\alpha)$ if and only if $\alpha_c|\, E = 0$. In the special Lagrangian case,

$$\alpha_c = \mathsf{Im}\left(dz^1 \wedge \cdots \wedge dz^n\right).$$

Thus an EDS for special Lagrangian submanifolds is $\mathcal{I} = \{\omega, \alpha_c\}_{\mathsf{alg}}$.

Given $E \in \mathcal{V}_n(\mathcal{I})$, we can again change coordinates so that E is annihilated by $dy^1, \dots, dy^n$. Again taking $e_1, \dots, e_n \in E$ to be dual to $dx^1, \dots, dx^n$, we have $s_1 = s_2 = \ldots = s_{n-2} = 1$. However,

$$\left.\begin{aligned} \omega &\equiv dx^{n-1} \wedge dy^{n-1} + dx^n \wedge dy^n \\ \alpha_c &\equiv dx^1 \wedge \cdots dx^{n-2} \wedge (dx^{n-1} \wedge dy^n - dx^n \wedge dy^{n-1}) \end{aligned}\right\} \mod dy^1, \dots, dy^{n-2}$$

shows that $s_{n-1} = 2$ (and $s_n = 0$). Since the requirement that $\alpha_c|\, E = 0$ is one additional equation on the set of Lagrangian n-planes, the fiber of $\mathcal{V}_n(\mathcal{I})$ has dimension $\binom{n+1}{2} - 1$, and the system is again involutive, with solutions depending on two functions of $n-1$ variables (thus recovering our observations from §5.12).

Exercise 7.5.6: Verify that the extra equation that special Lagrangian n-planes must satisfy is $\sum_j s^{jj} = 0$.

Example 7.5.7 (Associative submanifolds)**.** Recall that the 14-dimensional compact Lie group G_2 arises as the automorphism group of the algebra $\mathbb{O}$ of octonions (see §A.5), and leaves invariant a 3-form ϕ on $\mathbb{R}^7 = \mathsf{Im}\mathbb{O}$. This ϕ, which is defined by (A.11) in terms of the octonionic multiplication, is a calibration on $\mathbb{R}^7$, and its complement is $\phi_c = \frac{1}{2}\left|\mathsf{Im}\left((xy)z - (zy)x\right)\right|$ for $x, y, z \in \mathsf{Im}\mathbb{O}$. Since

$$\psi(x, y, z) := \mathsf{Im}\left((xy)z - (zy)x\right) = [x, y, z],$$

where $[x, y, z]$ denotes the associator (see Exercises A.5.3), we see that $\text{Face}(\phi)$ consists of 3-planes in $\mathbb{R}^7$ to which the octonionic multiplication restricts to be associative.

We define an EDS $\mathcal{I}$ for associative submanifolds by taking the components of the $\mathrm{Im}\mathbb{O}$-valued 3-form ψ as generators. (Since ψ is constant-coefficient, all of these generators are closed.) However, we can avoid a lengthy character calculation by using the fact that G_2 acts transitively on $\mathrm{Face}(\phi)$, as follows:

Choose the basis $e_1, \dots, e_7$ for $\mathbb{R}^7$ to coincide with the basis $\epsilon_1, \dots, \epsilon_7$ for $\mathrm{Im}\mathbb{O}$ in §A.5; then $\mathfrak{g}_2$ has the form (5.29). Without loss of generality we may assume that $E = \{e_1, e_2, e_3\} \in \mathcal{V}_3(\mathcal{I})$. Then the stabilizer of E in G_2 is six-dimensional. Thus, $V_3(\mathcal{I})$ is smooth of codimension four in $G(3, \mathbb{R}^7)$. On the other hand, for any flag in E, $c_0 = c_1 = 0$ and $c_2 = 4$ (two independent vectors in E determine the third one by multiplication). Thus, by Theorem 7.4.1 we have $\mathcal{I}$ involutive at E (hence involutive everywhere, by homogeneity). Integral manifolds depend on $s_2 = 4$ functions of two variables. (Note that this system has the same character and Cartan integer as the Pfaffian system for associative submanifolds discussed in §5.12.)

Exercises 7.5.8:

1. Recall that, if x is a point on a surface S in Euclidean space, then $x + (1/k_i)e_3$, $i = 1, 2$, is a point on one of the two *focal surfaces* of S. Show that, for the surfaces in Example 7.5.1, one of the focal surfaces degenerates to a curve. Show that the surfaces are, in fact, canal surfaces (see [**146**]) of radius $1/k_0$ about this curve, and use this to interpret the "two functions of one variable" in another way. ◎

2. Give an example of a surface whose principal curvatures k_1, k_2 are constant along their respective lines of curvature. Set up and investigate an EDS for these surfaces. (Hint: You should introduce k_1 and k_2 as new variables, and define your EDS on $\mathcal{F} \times \mathbb{R}^2$.) ◎

3. Show that, if we don't ignore Cauchy characteristics in applying Cartan's Test to Example 7.5.3, the system is still involutive but now $s_2 = 1$. Explain how the integral surfaces depend on one function of two variables. (Hint: How does the frame change as we move along the Cauchy characteristics?)

4. Set up and investigate an EDS for hypersurfaces in S^4 that have constant scalar curvature and are minimal—that is, the trace of the second fundamental form is zero. Can this be done without introducing the h_{ij} as new variables? ◎

5. Set up and investigate a non-Pfaffian EDS for hypersurfaces in S^5 that are Einstein manifolds [**11**]. Can something similar be set up for conformally flat hypersurfaces?

6. Set up and investigate a non-Pfaffian EDS for coassociative submanifolds of $\mathbb{E}^7$.

Chapter 8

Geometric Structures and Connections

We study the *equivalence problem* for geometric structures. That is, given two geometric structures (e.g., pairs of Riemannian manifolds, pairs of manifolds equipped with foliations, etc.), we wish to find differential invariants that determine existence of a local diffeomorphism preserving the geometric structures. We begin in §8.1 with two examples, 3-webs in the plane and Riemannian geometry, before concluding the section by defining G-structures. In order to find differential invariants, we will need to take derivatives in some geometrically meaningful way, and we spend some time (§§8.2–8.4) figuring out just how to do this. In §8.3 we define connections on coframe bundles and briefly discuss a general algorithm for finding differential invariants of G-structures. In §8.5 we define and discuss the holonomy of a Riemannian manifold. In §8.6 we present an extended example of the equivalence problem, finding the differential invariants of a path geometry. Finally, in §8.7 we discuss generalizations of G-structures and recent work involving these generalizations.

8.1. G-structures

In this section we present two examples of G-structures and then give a formal definition.

First example: 3-webs in $\mathbb{R}^2$.

First formulation of the question. Let $\mathcal{L} = \{L_1, L_2, L_3\}$ be a collection of three pairwise transverse foliations of an open subset $U \subseteq \mathbb{R}^2$. Such a structure is called a *3-web*; see Figure 1.

Let $\tilde{\mathcal{L}} = \{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\}$ be another 3-web on an open subset $\tilde{U} \subset \mathbb{R}^2$.

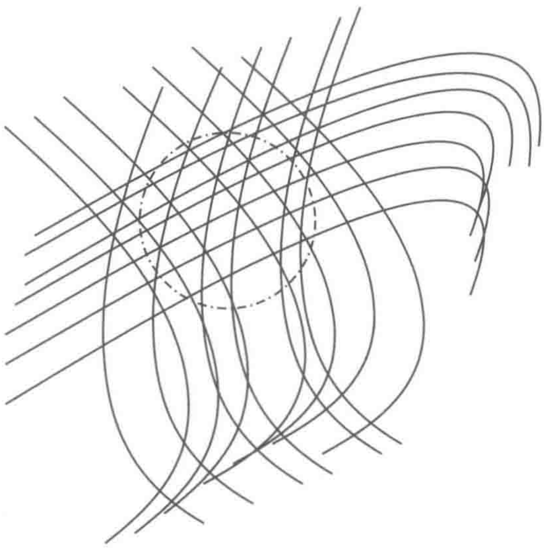


Figure 1. U is the region inside the circle

Problem 8.1.1. When does there exist a diffeomorphism $\phi : U \to \tilde{U}$ such that $\phi^*(\tilde{L}_j) = L_j$?

If there exists such a ϕ, we will say the webs $\mathcal{L}, \tilde{\mathcal{L}}$ are *equivalent*. We would like to find differential invariants that determine when two webs are equivalent, as we did for local equivalence of submanifolds of homogeneous spaces.

In particular, let $\mathcal{L}^0$ be the 3-web

$$L_1^0 = \{y = \text{const}\}, \quad L_2^0 = \{x = \text{const}\}, \quad L_3^0 = \{y - x = \text{const}\};$$

call this the *flat case*. We ask: when is a 3-web locally equivalent to the flat case?

Second formulation of the question. Let $y' = F(x, y)$ be an ordinary differential equation in the plane. Let $y' = \tilde{F}(x, y)$ be another.

Problem 8.1.2. When does there exist a change of coordinates $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\psi(x, y) = (\alpha(x), \beta(y))$ such that $\psi^* \tilde{F} = (\beta'/\alpha')F$? (I.e., so that solutions to one ODE are carried to solutions of the other.)

In particular, given F, is it equivalent to $y' = 1$ via a change of coordinates of the form of ψ?

Exercise 8.1.3: Determine local equivalence of first-order ordinary differential equations in the plane $y' = F(x, y)$ under arbitrary changes of coordinates. ◎

To see that Problems 8.1.1, 8.1.2 are the same, note that any two transverse foliations can be used to give local coordinates x, y, and the space of integral curves of an ODE in coordinates provides the third foliation. The diffeomorphisms of $\mathbb{R}^2$ that preserve the two coordinate foliations are exactly those of the form of ψ.

In order to study the local equivalence of webs, we would like to associate a coframe to a 3-web. For example, we could take a coframe $\{\underline{\omega}^1, \underline{\omega}^2\}$ such that

(a) $\underline{\omega}^1$ annihilates L_1,

(b) $\underline{\omega}^2$ annihilates L_2, and

(c) $\underline{\omega}^1 - \underline{\omega}^2$ annihilates L_3.

In the case of an ODE in coordinates, we could similarly take $\underline{\omega}^1 = F(x,y)dx, \underline{\omega}^2 = dy$.

Remark 8.1.4. Note that we are imitating the flat model ($\mathcal{L}^0$) on the infinitesimal level. This is what we did in Chapter 2 for Riemannian geometry when we took a basis of the cotangent space corresponding to the standard flat structure on the infinitesimal level. Just as any Riemannian metric looks flat to first order, so does any 3-web in the plane.

Just as in the case of choosing a frame for a submanifold of a homogeneous space, we need to determine how unique our choice of adapted frame is, and we will then work on the space of adapted frames.

Any other frame satisfying conditions (a) and (b) must satisfy

$$\begin{aligned}\underline{\tilde{\omega}}^1 &= \lambda^{-1}\underline{\omega}^1,\\ \underline{\tilde{\omega}}^2 &= \mu^{-1}\underline{\omega}^2\end{aligned}$$

for some nonvanishing functions λ, μ. Any frame satisfying (c) must be of the form

$$\underline{\tilde{\omega}}^2 - \underline{\tilde{\omega}}^1 = \nu^{-1}(\underline{\omega}^2 - \underline{\omega}^1).$$

Combining these three conditions, we see $\lambda = \mu = \nu$. Let $\mathcal{F}_{\mathcal{L}} \subset \mathcal{F}_{\mathsf{GL}}(U)$ be the space of coframes satisfying (a),(b),(c), a fiber bundle with fiber $\simeq \mathbb{R}^*$.

Dually, if we write a point of $\mathcal{F}_{\mathcal{L}}$ as a frame $f = (p, e_1, e_2)$, then e_1 is tangent to $L_{2,p}$, e_2 is tangent to $L_{1,p}$, and $e_1 + e_2$ is tangent to $L_{3,p}$.

Fixing a section $(\underline{\omega}^1, \underline{\omega}^2)$, we may use local coordinates (x, y, λ) on $\mathcal{F}_{\mathcal{L}}$. On $\mathcal{F}_{\mathcal{L}}$ we have tautological forms

$$\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} := \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}^{-1}\begin{pmatrix}\underline{\omega}^1\\ \underline{\omega}^2\end{pmatrix}.$$

Exercise 8.1.5: Show that these tautological forms are the pullbacks of the tautological forms of $\mathcal{F}_{\mathsf{GL}}(U)$. Thus, they are independent of our initial choice of $\underline{\omega}^1, \underline{\omega}^2$.

For submanifolds of homogeneous spaces, there was a canonical coframing of the space of adapted frames. Here, we have two 1-forms, but $\dim \mathcal{F}_{\mathcal{L}} = 3$ so we seek a third 1-form. When we faced the problem of completing a set of geometrically determined vectors to a basis before (as with submanifolds of homogeneous spaces), we differentiated. We do the same here:

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\frac{d\lambda}{\lambda^2} & 0\\ 0 & \frac{d\lambda}{\lambda^2}\end{pmatrix}\wedge\begin{pmatrix}\underline{\omega}^1\\ \underline{\omega}^2\end{pmatrix} + \lambda\begin{pmatrix}d\underline{\omega}^1\\ d\underline{\omega}^2\end{pmatrix}.$$

Since $\lambda d\underline{\omega}^j$ is semi-basic for the projection to $\mathbb{R}^2$, we may write $\lambda d\underline{\omega}^j = T^j\omega^1\wedge\omega^2$ for some functions $T^1, T^2 : \mathcal{F}_{\mathcal{L}} \to \mathbb{R}$. Write $\theta = \frac{d\lambda}{\lambda}$. Our equations now have the form

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + \begin{pmatrix}T^1\omega^1\wedge\omega^2\\ T^2\omega^1\wedge\omega^2\end{pmatrix}. \tag{8.1}$$

In analogy with the situation in §5.5, we will refer to the terms T^1, T^2 as "apparent torsion". More precisely, as we will see in §3, this is the *torsion of* θ. The forms $\omega^1, \omega^2, \theta$ give a coframing of $\mathcal{F}_{\mathcal{L}}$, but θ is not uniquely determined. In fact the choice of a θ satisfying (8.1) is unique up to modification by ω^1, ω^2: any other choice must be of the form $\tilde{\theta} = \theta + a\omega^1 + b\omega^2$. In particular, if we choose $\tilde{\theta} = \theta - T^2\omega^1 + T^1\omega^2$, our new choice has the effect that the apparent torsion is zero; there is a unique such form θ. So, renaming $\tilde{\theta}$ as θ, we have

Proposition 8.1.6. *There exists a unique form $\theta \in \Omega^1(\mathcal{F}_{\mathcal{L}})$ such that the equations*

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

are satisfied.

Any choice of θ such that the derivative of the tautological forms is of the form (8.1) will be called a *connection* (or connection form), and a choice of θ such that the torsion of θ is zero will be called a *torsion-free connection.*

The canonical coframing $(\omega^1, \omega^2, \theta)$ that we have constructed on $\mathcal{F}_{\mathcal{L}}$ enables us to begin to solve Problem 8.1.1:

Corollary 8.1.7. *Let $\phi : U \to U'$ be a diffeomorphism such that $\phi^*(\tilde{L}_j) = L_j$, and let $\Phi : \mathcal{F}_{\mathsf{GL}}(U) \to \mathcal{F}_{\mathsf{GL}}(\tilde{U})$ be the induced diffeomorphism on the coframe bundles (i.e., Φ takes a coframe to its pullback under ϕ^{-1}). Then $\Phi(\mathcal{F}_{\mathcal{L}}) = \Phi(\mathcal{F}_{\tilde{\mathcal{L}}})$ and $\Phi^*(\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\theta}) = (\omega^1, \omega^2, \theta)$.*

Exercise 8.1.8: Prove this corollary.

Moreover, further necessary conditions come from differential invariants, which we obtain by differentiating the canonical coframe. To calculate $d\theta$, we compute

$$0 = d^2\omega^1 = -d\theta \wedge \omega^1 - \theta \wedge (\theta \wedge \omega^1) = -d\theta \wedge \omega^1,$$
$$0 = d^2\omega^2 = -d\theta \wedge \omega^2 - \theta \wedge (\theta \wedge \omega^2) = -d\theta \wedge \omega^2.$$

If $\alpha \in \Omega^1(\mathcal{F}_{\mathcal{L}})$ is any form, then $d\alpha = A\theta \wedge \omega^1 + B\theta \wedge \omega^2 + C\omega^1 \wedge \omega^2$, for some functions $A, B, C : \mathcal{F}_{\mathcal{L}} \to \mathbb{R}$. Since $d\theta \wedge \omega^j = 0$, we see that

$$d\theta = K\omega^1 \wedge \omega^2$$

for some function $K : \mathcal{F}_{\mathcal{L}} \to \mathbb{R}$.

Exercise 8.1.9: Show that the form $d\theta$ is basic (i.e., well-defined on U). It is called the *Blaschke-Chern curvature form* of the web.

Exercise 8.1.10: Calculate the Blaschke-Chern curvature form for the following webs:

(1) $\{x = \text{const}, y = \text{const}, x - Cy = \text{const}\}$, where C is a constant.
(2) $\{x = \text{const}, y = \text{const}, x/y = \text{const}\}$.
(3) $\{x = \text{const}, y = \text{const}, x^n + y^n = \text{const}\}$.
(4) $\{x = \text{const}, y = \text{const}, F(x, y) = \text{const}\}$ (the general case).

Exercise 8.1.9 implies that while the function K is not well-defined on U, the property $K = 0$ is.

Remark 8.1.11 (Hexagonality). Recall that one can interpret the Gauss curvature of a surface in $\mathbb{E}^3$ in terms of a limit of ratios of areas. There is a similar geometric interpretation of the Blaschke-Chern curvature, in terms of the following construction (see Figure 2):

Fix $x \in U$. Travel along a leaf of L_1 to a nearby point a, then along L_3 to a point b on the L_2-leaf through x. Now travel along L_1 until you hit the leaf of L_3 that passes through x. Then travel again along L_2 until you hit the leaf of L_1 that passes through x, then along L_3 until you hit the leaf of L_2 that passes through x, then along L_1 until you hit the leaf of L_3 passing through x, and finally along L_2 until you hit the leaf of L_1 passing through x finishing at a point g. The path $a - g$ traced out is necessarily closed iff $K \equiv 0$ (so, in the case of the standard flat model, the figure will be a hexagon). Moreover, the failure of the figure to close up is measured by K; see [**124**], p.164.

Second example: Riemannian geometry. Let (M^n, g) be a Riemannian manifold. We have already seen that the natural space of adapted frames is the orthonormal frames.

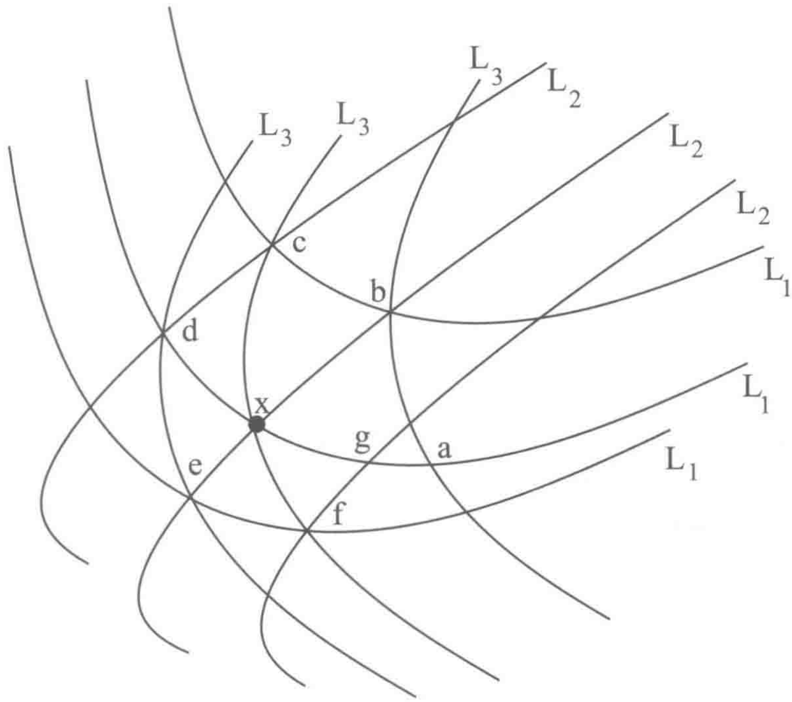


Figure 2. a non-hexagonal 3-web

Let $(\underline{\omega}^1, \ldots, \underline{\omega}^n)$ be a local orthonormal coframing, i.e., a section of the bundle of orthonormal frames of M which we denote $\mathcal{F}_{on}$. Any other orthonormal coframing is of the form

$$\begin{pmatrix} \underline{\tilde{\omega}}^1 \\ \vdots \\ \underline{\tilde{\omega}}^n \end{pmatrix} = a(x)^{-1} \begin{pmatrix} \underline{\omega}^1 \\ \vdots \\ \underline{\omega}^n \end{pmatrix}$$

with $a(x) \in O(n)$.

Locally we may specify a point of $\mathcal{F}_{on}$ by a pair (x, a), where $x : M \to \mathbb{R}^n$ is a local coordinate on M. Let $\omega_{x,a} := \pi^*(a^{-1}\underline{\omega}_x)$.

We calculate

$$d\omega = -a^{-1}da \wedge a^{-1}\underline{\omega}_x + a^{-1}d\underline{\omega}_x.$$

Let $\theta^j_k = (a^{-1}da)^j_k$. Observe that $a^{-1}d\underline{\omega}_x$ is semi-basic and the forms $\omega^i \wedge \omega^j$ span the semi-basic 2-forms, so we may write

$$d\omega^j = -\theta^j_k \wedge \omega^k + T^j_{kl}\omega^k \wedge \omega^l, \tag{8.2}$$

where the 1-forms θ^i_j satisfy $\theta^j_k + \theta^k_j = 0$ and the functions T^i_{jk} satisfy $T^j_{kl} = -T^j_{lk}$. Without indices, (8.2) can be written as

$$d\omega = -\theta \wedge \omega + \tfrac{1}{2}T(\omega \wedge \omega),$$

where $V \simeq \mathbb{R}^n$, $\theta \in \Omega^1(\mathcal{F}_{on}, \mathfrak{so}(V))$ and $T(\omega \wedge \omega) \in C^\infty(\mathcal{F}_{on}, \Lambda^2 V^* \otimes V)$.

We now prove that it is possible to choose a unique connection $\theta \in \Omega^1(\mathcal{F}_{on}, \mathfrak{so}(V))$ such that the $T^i_{jk} = 0$ for all i, j, k, i.e., the torsion of the connection θ is zero.

First note that any change in connection form must be of the form

$$\tilde{\theta}^i_j = \theta^i_j + A^i_{jk}\omega^k$$

for some functions A^i_{jk}. In order that $\tilde{\theta}$ be $\mathfrak{so}(V)$-valued, we need $A^i_{jk} = -A^j_{ik}$. To express this without indices, consider $A = A^i_{jk} v_i \otimes v^j \otimes v^k \in V \otimes V^* \otimes V^*$, where we fix a basepoint and let $\{v_i\}$ be a basis of V with dual basis $\{v^i\}$. Then, we are requiring that $A \in \mathfrak{so}(V) \otimes V^* \subset V \otimes V^* \otimes V^*$.

Now, the change any $A \in V \otimes V^* \otimes V^*$ effects is $\tilde{T}^i_{jk} = T^i_{jk} + (A^i_{jk} - A^i_{kj})$, so we let $\delta : V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$ denote the skew-symmetrization map. We see that the change A effects is $\delta(A)$. To show existence of a torsion-free connection, we need to show that the restriction of δ to $\mathfrak{so}(V) \otimes V^*$ surjects onto $V \otimes \Lambda^2 V^*$; then any apparent torsion can be absorbed.

Exercise 8.1.12: The restriction of δ to $\mathfrak{so}(V) \otimes V^*$ is injective. ◎

By counting dimensions, $\delta(\mathfrak{so}(V) \otimes V^*) = V \otimes \Lambda^2 V^*$. Thus, a torsion-free connection exists and is unique. (This proves the upstairs version of the fundamental Lemma 2.6.8 of Riemannian geometry.)

In Chapter 2, to find differential invariants of Riemannian metrics we computed $0 = d^2\omega$. We saw that $0 = d^2\omega = (d\theta + \theta \wedge \theta) \wedge \omega$ and that $\Theta := d\theta + \theta \wedge \theta \in \Omega^2(M, \mathfrak{so}(TM))$ is a differential invariant, called the *Riemann curvature tensor*. Note that Θ, like most of the differential invariants, is vector bundle valued.

Definition of G-structures. We want a definition of a G-structure on a manifold M that generalizes our two examples. First we generalize the example of Riemannian geometry:

Let $V = \mathbb{R}^n$ with its standard basis $\{e_i\}$, and $GL(V) = GL(n, \mathbb{R})$. Let $\mathbb{T}(V)$ be a tensor space of V, e.g., $\mathbb{T}(V) = S^2V^*$, $\mathbb{T}(V) = \Lambda^2 V^*$ or $\mathbb{T}(V) = V^* \otimes V$. Let $\psi \in \mathbb{T}(V)$ be a tensor and let $G = G_\psi \subset GL(V)$ be the subgroup preserving ψ. (In other words, if $\rho : G \to GL(\mathbb{T}(V))$ is the induced action of G on $\mathbb{T}(V)$, then $g \in G$ if and only if $\rho(g)\psi = \psi$.) For example,

$$Q \in S^2V^*,$$
$$\omega \in \Lambda^2 V^*,$$
$$J \in V^* \otimes V \text{ such that } J^2 = -\mathrm{Id}$$

respectively define the orthogonal group $G_Q = O(V, Q)$, the symplectic group $G_\omega = Sp(V, \omega)$, and the complex linear group $G_J = Gl(m, \mathbb{C})$ for $m = 2n$. (Q and ω are assumed to be nondegenerate.)

Recall that over any differentiable manifold M^n we have the general coframe bundle $\pi : \mathcal{F}_{\mathsf{GL}}(M) \to M$ whose points are $f = (x, u)$, where $x \in M$ and $u : T_xM \to V$ is a linear isomorphism. We define a right action of $GL(V)$ on the fibers of $\mathcal{F}_{\mathsf{GL}}$ by

$$(x, u) \cdot g := (x, g^{-1} \circ u). \tag{8.3}$$

Given $\psi \in \mathbb{T}(V)$, let

$$\tilde{\psi} \in \Gamma(M, \mathbb{T}(TM))$$

be a smooth section such that at each point $x \in M$, there exists a coframe u for which the induced map $u^{\mathbb{T}} : \mathbb{T}(T_xM) \to \mathbb{T}(V)$ takes $\tilde{\psi}$ to ψ. (For example, if $\psi \in S^2V^*$ is an inner product, then $\tilde{\psi}$ is a Riemannian metric and u is an orthonormal coframe.) Using $\tilde{\psi}$, we define a reduction of $\mathcal{F}_{\mathsf{GL}}$ to a subbundle

$$\mathcal{F}_G = \{(x, u) \in \mathcal{F}_{\mathsf{GL}} \mid u^{\mathbb{T}}(\tilde{\psi}) = \psi\}.$$

In other words, the tensor field $\tilde{\psi}$ enables us to define a class of preferred coframes adapted to the geometry induced by $\tilde{\psi}$. For example, taking ψ to be Q, ω or J as above makes M into a *semi-Riemannian manifold*, an *almost symplectic manifold* or an *almost complex manifold*. (A semi-Riemannian manifold is Riemannian if the quadratic form is positive definite; almost complex manifolds are discussed in Example 8.3.14 and Appendix C.) Although our first example, web geometry, is not naturally described as arising from a tensor, it can be described in terms of the reduction of the frame bundle, so we take that as our definition of a G-structure:

Definition 8.1.13. A *G-structure* on a differentiable manifold M is a reduction of $\mathcal{F}_{\mathsf{GL}}$ to a right G-bundle $\mathcal{F}_G \subset \mathcal{F}_{\mathsf{GL}}$, where $G \subset GL(V)$ is a matrix Lie group.

(In order to be a right G-bundle, G must have a smooth right action whose restriction to the fibers is simply transitive, i.e., any two points in the same fiber are related by a unique element of G.) Sometimes we will abbreviate $\mathcal{F}_G$ by B.

Any subgroup $G \subset GL(V)$ acts on $\mathcal{F}_{\mathsf{GL}}$ by restriction of (8.3), so we may form the quotient bundle $\mathcal{F}_{\mathsf{GL}}/G$ featured in the following alternative definition:

Definition 8.1.14. A G-structure on M is a smooth section $s : M \to \mathcal{F}_{\mathsf{GL}}/G$.

To see that this is equivalent to our previous definition, let $\pi_G : \mathcal{F}_{\mathsf{GL}} \to \mathcal{F}_{\mathsf{GL}}/G$ denote the projection; then $\mathcal{F}_G = \pi_G^{-1}(s)$. In the other direction, $\pi_G(\mathcal{F}_G)$ is a section of $\mathcal{F}_{\mathsf{GL}}/G$.

Given a diffeomorphism $\phi : M \to M'$, we obtain an identification $\mathcal{F}_{\mathsf{GL}}(M) \simeq \mathcal{F}_{\mathsf{GL}}(M')$ as follows: First we may pull back $\mathcal{F}_{\mathsf{GL}}(M')$ to obtain a bundle $\phi^*(\mathcal{F}_{\mathsf{GL}}(M')) \to M$. Next, we use composition with $\phi_* : TM \to TM'$ to identify points of $\phi^*(\mathcal{F}_{\mathsf{GL}}(M'))$ with coframes of M, and finally we observe that all these identifications are smooth and natural. We call this identification the *induced identification* of $\mathcal{F}_{\mathsf{GL}}(M)$ with $\mathcal{F}_{\mathsf{GL}}(M')$. If $G \subset GL(V)$, then, using ϕ, we similarly get an induced identification of $\mathcal{F}_{\mathsf{GL}}(M)/G$ with $\mathcal{F}_{\mathsf{GL}}(M')/G$.

Definition 8.1.15. Two G-structures $\mathcal{F}_G \subset \mathcal{F}_{\mathsf{GL}}(M)$ and $\mathcal{F}'_G \subset \mathcal{F}_{\mathsf{GL}}(M')$ are *equivalent* if there exists a diffeomorphism $\phi : M \to M'$ such that the induced identification of $\mathcal{F}_{\mathsf{GL}}(M)$ with $\mathcal{F}_{\mathsf{GL}}(M')$ takes $\mathcal{F}_G$ to $\mathcal{F}'_G$. In other words, $s : M \to \mathcal{F}_{\mathsf{GL}}(G)/G$ and $s' : M \to \mathcal{F}_{\mathsf{GL}}(M')/G$ are equivalent if and only if there exists a diffeomorphism $\phi : M \to M'$ such that under the induced identification of $\mathcal{F}_{\mathsf{GL}}(M)/G$ with $\mathcal{F}_{\mathsf{GL}}(M')/G$, we have $s = s'$.

Exercise 8.1.16: Prove that $\mathcal{F}_G$ and $\mathcal{F}'_G$ are equivalent if and only if there exists a diffeomorphism $\Phi : \mathcal{F}_G \to \mathcal{F}'_G$ such that $\Phi^*(\omega') = \omega$.

Definition 8.1.17. A G-structure is *flat* if for all $x \in M$ there exist local coordinates $x^1, \dots, x^n$ on a neighborhood of x such that $(dx^1, \dots, dx^n)$ is a local section of $\mathcal{F}_G$.

For example, an $O(n)$-structure (equivalently, a Riemannian metric) is flat iff there exist local coordinates $x^1, \dots, x^n$ such that dx^i gives a local orthonormal coframing of M, i.e., coordinates such that $g = (dx^1)^2 + \dots + (dx^n)^2$.

Exercise 8.1.18: Show that an $Sp(2m, \mathbb{R})$-structure determined by a non-degenerate 2-form ω is flat if there exist local coordinates $(x^1, \dots, x^{2m})$ such that in those coordinates, $\omega = dx^1 \wedge dx^{m+1} + \dots + dx^m \wedge dx^{2m}$.

8.2. How to differentiate sections of vector bundles

In order to find differential invariants, we will need to be able to take derivatives, not just of functions on M, but of sections of vector bundles over M as well.

Given a manifold M and a vector-valued function $f : M \to \mathbb{R}^r$, we can calculate $df \in \Omega^1(M, \mathbb{R}^r)$ as follows: choose a basis $v_1, \dots, v_r$ of $\mathbb{R}^r$, write $f = f^i v_i$ and let $df = df^i \otimes v_i$. If we choose another basis $\tilde{v}_i = a^j_i v_j$, then in this basis $f = \tilde{f}^i \tilde{v}_i = (f^i (a^{-1})^j_i) \tilde{v}_j$. In our new basis $df = (df^i (a^{-1})^j_i) \otimes \tilde{v}_j =$

$df^i \otimes v_i$ and thus df is well-defined. In this way we can take successive derivatives, for if $X \in T_xM$, then $df(X)$ is again a vector-valued function and so, if X is extended to a vector field, for $Y \in T_xM$, we may take $Y(X(f))$, etc.

Now let $\pi : E \to M$ be a rank r vector bundle. We would like to differentiate sections $s : M \to E$. Since E is a differentiable manifold, we have $ds_x : T_xM \to T_{s(x)}E$, i.e., $ds \in C^\infty(TM, E)$. We would like a derivative whose values at $x \in M$ are in the fiber $E_x = \pi^{-1}(x)$, just as in our previous example where $E = V \times M$ was the trivial bundle. Let's use the notation $\nabla s \in \Omega^1(M, E)$ for the object we are looking for. We'll try to find ∇s in two different ways.

First try. Locally, over contractible open sets $U \subset M$, $E|_U$ is trivial, so after choosing a trivialization, our above basis definition works. The problem is on the overlap of two such neighborhoods, $U \cap V$, we may need to use a change of basis that depends on $x \in M$. Say on U we have $s = s^i v_i$ and on V we have $s = \tilde{s}^i \tilde{v}_i$. On $U \cap V$ we will have $\tilde{v}^i = a_i^j(x) v_j$. As a first try at differentiating s, write $\nabla^U s = ds^i \otimes v_i$ and $\nabla^V s = d\tilde{s}^i \otimes \tilde{v}_i$. Then

$$\begin{aligned}
\nabla^V s &= d\tilde{s}^i \otimes \tilde{v}_i \\
&= d(s^i (a^{-1})_i^j) \otimes \tilde{v}_j \\
&= ds^i \otimes (a^{-1})_i^j \tilde{v}_j + s^i d((a^{-1})_i^j) \otimes \tilde{v}_j \\
&= ds^i \otimes v_i - f^i (a^{-1} da)_i^j \otimes v_j \\
&= \nabla^U s - (a^{-1} da) s.
\end{aligned}$$

The error term $\nabla^U s - \nabla^V s = (a^{-1} da) s$ looks suspiciously like a Maurer-Cartan form. We will be interested in the situation when M is equipped with a G-structure and the fibers of E are acted on by G (see 8.4 for how this action is defined). In this case we will see that the function a is G-valued, and the presence of the $a^{-1} da$ term will be a measure of the failure of the G-structure to be flat.

What we will do to eliminate the error term, and make the derivatives of s agree on the overlaps, is to define $\nabla^U s = ds^i \otimes v_i + \mathrm{stuff}_U$ in a way so that $\mathrm{stuff}_U - \mathrm{stuff}_V = -(a^{-1} da) s$.

Second try. Recall that there is a distinguished subspace of T_pE, the *vertical subspace*

$$\mathcal{V}_p E := \{v \in T_pE \mid \pi_{*p}(v) = 0\},$$

and that $\mathcal{V}_p E$ is naturally isomorphic to the fiber $E_{\pi(p)}$. So, if we could define a projection $\mathrm{proj}_{\mathsf{vert}} : T_pE \to \mathcal{V}_p E \simeq E_{\pi(p)}$, we could define

$$\nabla s := \mathrm{proj}_{\mathsf{vert}} \circ ds. \tag{8.4}$$

Given a vector space V and a subspace $W \subset V$, to define a projection onto W we need a complementary subspace W^c such that $V = W \oplus W^c$. Then any vector $v \in V$ can be written uniquely as $v = v_1 + v_2$ such that $v_1 \in W, v_2 \in W^c$, and the projection is $v \mapsto v_1$. But in general we don't have any naturally defined complement to $\mathcal{V}E$. In the very special case of Riemannian geometry, when $E = TM$, there is the Riemannian metric induced on $T(TM)$ that ensures such a complement, and one obtains a well-defined operator ∇.

Using this projection idea, we may simply define

Definition 8.2.1. A *connection* on a vector bundle E is a choice of complement to $\mathcal{V}E$. A connection induces a *covariant differential operator* $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ as in (8.4).

If $X \in \Gamma(TM)$, we write $\nabla_X : \Gamma(E) \to \Gamma(E)$ for the induced differential operator. We note that such a ∇ has the following properties:

1. $\nabla(s+t) = \nabla s + \nabla t$ for all $s, t \in \Gamma(E)$.
2. $\nabla(fs) = df \otimes s + f \nabla s$ for all $s \in \Gamma(E)$, $f \in C^\infty(M)$.
3. For $X, Y \in \Gamma(TM)$, with $X_p = Y_p$ at some $p \in M$, $(\nabla_X)_p = (\nabla_Y)_p$.

Exercises 8.2.2:

1. Let $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ be any operator satisfying properties 1,2 and 3 above. Let x^i be local coordinates on M and v_α local linear coordinates in the fiber of E, so $s : M \to E$ may be written as $s(x) = s^\alpha(x) v_\alpha$. Show that $\nabla s = ds^\alpha \otimes v_\alpha + s^\alpha \Gamma^\beta_{\alpha,i} dx^i \otimes v_\beta$ for some functions $\Gamma^\beta_{\alpha,i}(x)$. The functions $\Gamma^i_{\alpha,\beta}$ are called the *Christoffel symbols* for ∇.
2. Using part one, show that any operator ∇ satisfying 1,2,3 gives rise to a connection on E. What is the complement to $\mathcal{V}E$ in coordinates?

We now have a structure that enables us to differentiate sections of vector bundles. This will be reconciled with our first attempt in §8.4 below.

It remains to see how to define such a structure in a way that is compatible with a group action on the fibers of E. Before going into details, let's look at a familiar case:

Example 8.2.3. Let G be a matrix Lie group, and let $M^n = G/H$ be a homogeneous space, with quotient map $\pi : G \to M$. Assume that if we write $g \in G$ as $g = (x, e_1, \ldots, e_n)$, where x and $e_1, \ldots, e_n$ are the columns of the matrix representing g, the projection is $\pi(g) = x$ and we may identify $e_1, \ldots, e_n$ with a basis of T_xM. (This was the situation in many of our examples in Chapters 1–3.) Assume that there exists a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is H-invariant. (This will be the case, e.g., if H is reductive.). Then we may write $de_j = e_i \otimes \omega^i_j$, where the ω^i_j are components of the Maurer-Cartan form of H.

Let $s : M \to G$ be a section and for $X = x^i s^*(e_i) \in \Gamma(TM)$ define

$$\begin{aligned}\nabla X &= dx^i {\otimes} s^*(e_i) + x^i s^*(de_i)\\ &= (dx^j + x^i s^*(\omega_i^j)){\otimes} s^*(e_j).\end{aligned}$$

To see ∇X is well-defined, let $\tilde{s}(x) = s(x)a(x)$ be another section where $a : M \to G$, so $\tilde{s}^*(e_i) = a_i^j s^*(e_j)$. Recall from (1.19) that

$$\tilde{s}^*(\omega_j^i) = (a^{-1}da + a^{-1}s^*(\omega)a)_j^i.$$

Exercises 8.2.4:

1. Finish the calculation to see that ∇X is independent of the choice of s.
2. Generalize the above example to the case where G is not necessarily a matrix Lie group.

Note that here the Christoffel symbols Γ_{jk}^i for the connection are defined by $s^*(\omega_j^i) = \Gamma_{jk}^i s^*(\omega^k)$. To relate this example to our first attempt, the correction term we are adding on to the naïve derivative $dx^i \otimes e_i$ is the pullback of the Maurer-Cartan form of $\mathfrak{h}$. Moreover, the choice of different sections is analogous to different local trivializations, and we see that we do indeed get the desired cancellation.

8.3. Connections on $\mathcal{F}_G$ and differential invariants of G-structures

Inspired by Example 8.2.3 for homogeneous spaces, we guess that having a coframing of $\mathcal{F}_G$ might enable us to define a G-equivariant connection on TM. Also, if we had such a coframing, derivatives of the coframing would furnish us with differential invariants.

The coframe bundle $\mathcal{F}_{\mathsf{GL}}$ comes equipped with—and $\mathcal{F}_G$ inherits—a tautological V-valued 1-form ω defined by

$$\omega_{(x,u)}(w) = u(\pi_*(w)).$$

This furnishes a basis $\{\omega^i\}$ for the semi-basic forms on $\mathcal{F}_G$ by writing $\omega := \omega^i e_i$.

To obtain a coframing of $\mathcal{F}_G$ we need to find a complementary set of forms to the $\{\omega^i\}$. We want to mimic the situation of frames for submanifolds of homogeneous spaces $M = K/G$ as much as possible. In that case, the complement to the set of semi-basic forms (i.e., the $\mathfrak{m}$-valued part of the Maurer-Cartan form of K) was a $\mathfrak{g}$-valued 1-form. (Remember the fiber of $K \simeq \mathcal{F}_G \to M$ is isomorphic to G.) For an arbitrary G-structure, the fiber of $\pi : \mathcal{F}_G \to M$ is also isomorphic to G, so we seek a complement to the ω^i that is $\mathfrak{g}$-valued. To imitate the Maurer-Cartan form as much as possible, we will ask for a form that "pulls back" to be the Maurer-Cartan form in the sense of the following definition:

Definition 8.3.1. Let $\underline{\theta} \in \Omega^1(G, \mathfrak{g})$ denote the Maurer-Cartan form of G. If we fix a point u_0 in a fiber $\mathcal{F}_{G,x}$, then there is a mapping $\mu_{u_0} : G \to \mathcal{F}_{G,x}$ defined by $g \mapsto u_0 \cdot g$.

A *connection form* is a $\mathfrak{g}$-valued 1-form θ on $\mathcal{F}_G$, i.e., $\theta \in \Omega^1(\mathcal{F}_G, \mathfrak{g})$, such that for all $x \in M$ and for some $u_0 \in \mathcal{F}_{G,x}$, we have $\mu_{u_0}^*(\theta) = \underline{\theta}$.

Exercises 8.3.2:
1. Show that if θ is a connection form, then $T^*\mathcal{F}_G$ is spanned by the entries of θ and ω.
2. Show that if $u_1 \in \mathcal{F}_{G,x}$ is another element, that $\mu_{u_0}^*(\theta) = \mu_{u_1}^*(\theta)$, so the definition does not depend on such choices.

Proposition 8.3.3. *$\theta \in \Omega^1(\mathcal{F}_G, \mathfrak{g})$ is a connection form if and only if it satisfies the structure equation*

$$d\omega = -\theta \wedge \omega + \tfrac{1}{2}T(\omega \wedge \omega), \tag{8.5}$$

where $T_{(x,u)} : \Lambda^2 V \to V$ is a linear map, and we use the notation $\omega \wedge \omega(v, w) := \omega(v) \wedge \omega(w)$, so that $\omega \wedge \omega \in \Omega^2(\mathcal{F}_G, \Lambda^2 V)$. In indices, (8.5) can be written as

$$d\omega^i = -\theta^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k,$$

where $\theta^i_j v_i \otimes v^j \in \Omega^1(\mathcal{F}_G, \mathfrak{g})$ and $T^i_{jk} = -T^i_{kj}$.

Proof. It is sufficient to work locally; let $(x^1, \dots, x^n)$ be coordinates on an open subset $U \subset M$. We have $dx \in \Gamma(M, \mathcal{F}_{\mathsf{GL}})$, and there exists a map $h : U \to GL(V)$ such that $\underline{\omega} = h^{-1}dx \in \Gamma(U, \mathcal{F}_G)$. Write $(x, h^{-1}dx) = (x, u_0(x))$. Then $\mathcal{F}_G \mid_U \simeq U \times G$ with $(\mathcal{F}_G)_x = \{(x, u_0 \cdot a) \mid a \in G\}$.

Note that $\mu_{u_0}^*(a^{-1}da) = \underline{\theta}$, so $a^{-1}da$ is a connection form. We calculate

$$\begin{aligned} d\omega &= -a^{-1}daa^{-1}h^{-1} \wedge dx - ah^{-1}dhh^{-1} \wedge dx \\ &= -(a^{-1}da) \wedge \omega - a^{-1}h^{-1}dhh^{-1} \wedge dx. \end{aligned}$$

The second term is semi-basic; thus it can be written as a linear combination of the $\omega^i \wedge \omega^j$, so our connection $a^{-1}da$ satisfies the structure equation (8.5). Moreover, any other connection is of the form $a^{-1}da + \beta$ with $\beta \in \Omega^1(\mathcal{F}_G, \mathfrak{g})$ semi-basic. Hence $\mu_{u_0}^*(\beta) = 0$, so $a^{-1}da + \beta$ still satisfies the structure equation (8.5).

In the other direction, we see that any $\mathfrak{g}$-valued form satisfying (8.5) must be of the form $a^{-1}da + \beta$. □

Definition 8.3.4. $T_\theta := T^i_{jk} v_i \otimes v^j \wedge v^k \in \Gamma(\mathcal{F}_G, V \otimes \Lambda^2 V^*)$ is called the *torsion of the connection* θ, and will sometimes be denoted simply by T. The analogue of T_θ in the context of linear Pfaffian systems was called apparent torsion.

We are less interested in T_θ than in the part of the torsion that is independent of our choice of θ. Recall from §5.5 that the torsion of a linear Pfaffian system with tableau $A \subseteq W \otimes V^*$ lies in

$$H^{0,2}(A) = (W \otimes \Lambda^2 V^*)/\delta(A \otimes V^*),$$

where $\delta : (W \otimes V^*) \otimes V^* \to W \otimes \Lambda^2 V^*$ is the skew symmetrization map. Now let $W = V$ and $A = \mathfrak{g}$.

Definition 8.3.5. Let s be a G-structure on M, θ a connection form and $T_\theta \in \Gamma(\mathcal{F}_G, V \otimes \Lambda^2 V^*)$ the torsion of θ. Then $[T_\theta] \in \Gamma(\mathcal{F}_G, H^{0,2}(\mathfrak{g}))$ is called the *torsion of the G-structure*. A G-structure with zero torsion is called *1-flat*, because it resembles a flat G-structure to first order.

Exercises 8.3.6:
1. If $\mathfrak{g} \subseteq \mathfrak{so}(V)$, then $H^{0,2}(\mathfrak{g}) = (\mathfrak{so}(V)/\mathfrak{g}) \otimes V^*$.
2. Verify that $[T_\theta]$ is independent of θ.
3. If $\mathcal{F}_G \to M$ is any G-structure, then for all $x \in M$, there exist local coordinates (x^i) such that, *at* x, $(dx^1, \ldots, dx^n) \in (\mathcal{F}_G)_x$. If $\mathcal{F}_G$ is 1-flat, then moreover, we can have that, *at* x, $\theta_x = 0$. (But of course we cannot have $d\theta_x = 0$ unless $\mathcal{F}_G$ is *2-flat* at x; see 8.3.10 below.)

To determine the uniqueness of a choice of connection with given torsion (in particular, zero torsion), we need to calculate the changes of connection that do not alter the torsion. This is the prolongation:

$$\mathfrak{g}^{(1)} = \ker \delta = (\mathfrak{g} \otimes V^*) \cap (V \otimes S^2 V^*).$$

We saw in §5.5 that the prolongation of a tableau A admits the interpretation as the modifications of the forms π^a_i, which in this context are the entries of the connection form, that do not affect the torsion.

Remark 8.3.7. We may rephrase the fundamental lemma of Riemannian geometry as

$$H^{0,2}(\mathfrak{so}(V)) = (V^* \otimes \Lambda^2 V)/\delta(\mathfrak{so}(V) \otimes V) = 0$$

and

$$\mathfrak{so}(V)^{(1)} = (\mathfrak{so}(V) \otimes V^*) \cap (V \otimes S^2 V^*) = 0,$$

which respectively show existence and uniqueness of a torsion-free connection.

When ρ is a G-structure with $[T] = 0$ and $\mathfrak{g}^{(1)} = 0$, we calculate $0 = d^2\omega = (d\theta + \theta \wedge \theta) \wedge \omega$ and observe that $R := d\theta + \theta \wedge \theta \in \Omega^2(\mathcal{F}_G, \mathfrak{g})$ is a well-defined differential invariant, called the *curvature* of the G-structure. If $\mathfrak{g}^{(1)} \neq 0$ and θ is a torsion-free connection, then we may still define R_θ, but now it is just the curvature of θ.

Exercise 8.3.8: Using Exercise A.1.11, show that R is semi-basic. We will see in Exercise 8.4.6 that if $\mathfrak{g}^{(1)} = 0$ then R is actually basic and descends to give a well-defined vector bundle valued 2-form on M.

Exercise 8.3.9 (The first Bianchi identity)**:** Let $\delta_2 : (V{\otimes}V^*){\otimes}\Lambda^2 V^* \to V{\otimes}\Lambda^3 V^*$ denote the skew-symmetrization map. Show that

$$R_{\theta,f} \in (\mathfrak{g}{\otimes}\Lambda^2 V^*) \cap (V{\otimes}S_{21}V^*) = \ker\, \delta_2|_{\mathfrak{g}\otimes\Lambda^2 V^*}.$$

Remark 8.3.10. If s is a flat G-structure on M, then there exists a connection θ with $T_\theta = 0$ and $R_\theta = 0$. A G-structure such that there exists a connection θ such that $T_\theta = 0$ and $R_\theta = 0$ is called 2-*flat*, because at each point the structure appears flat to order two.

Remark 8.3.11 (Flat G-structures). The Goldschmidt version of the Cartan-Kähler Theorem 5.7.5 has the corollary that $H^{k,2}(\mathfrak{g})$ contains all the obstructions to flatness for a G-structure. Say all but a finite number of the $H^{k,2}(\mathfrak{g})$ are zero. Then, to check if a given G-structure is flat, it is only necessary to calculate the invariants taking value in the nonzero $H^{k,2}(\mathfrak{g})$.

Given a group G, how do we compute $H^{i,j}(\mathfrak{g})$? If $\mathfrak{g}$ has a nice representation theory (e.g., $\mathfrak{g}$ is semi-simple), then the Spencer cohomology groups can be computed using the *Bott-Borel-Weil theorem* (see, e.g., [**154, 62**]). For example, for a certain very nice class of representations of simple Lie algebras (the sub-minuscule ones, those occurring as the isotropy representation on the tangent space of a compact Hermitian symmetric space) Goncharov showed that there is exactly one nonzero $H^{j,2}$; see §8.7.

The algorithm for solving the equivalence problem is due to Cartan. In the next two subsections we will mention some steps in this algorithm, to give the reader an idea of its flavor. This algorithm is presented with more detail in [**54**] and [**139**]. Rather than attempting to give an entire flowchart with lots of new definitions, we will compute one example in gory detail in §8.6 that will hopefully give the reader enough of an idea what to do for any particular G-structure problem she or he might come across.

Prolongation: What to do if $\mathfrak{g}^{(1)} \neq 0$. We eliminate the ambiguity in choice of connection form by working on a larger space, the enlargement being the addition of variables that parametrize the choice of connection form. (This enlargement is analogous to our working on an adapted coframe bundle instead of choosing a particular adapted coframe.) Let $p^1 : {\mathcal{F}_G}^{(1)} \to \mathcal{F}_G$ be the principal bundle whose fiber is the additive group $\mathfrak{g}^{(1)}$. More precisely, let $p : \mathcal{F}(\mathcal{F}_G) \to \mathcal{F}_G$ denote the coframe bundle of $\mathcal{F}_G$, where we write a point of $\mathcal{F}(\mathcal{F}_G)$ by (x, u, ψ). Then ${\mathcal{F}_G}^{(1)} \subset \mathcal{F}(\mathcal{F}_G)$ is the subbundle where $\psi = (p^*(\omega), p^*(\theta))$, where θ is a connection form at (x, u). Note that the fiber of p^1 is $\mathfrak{g}^{(1)}$.

We continue to differentiate on ${\mathcal{F}_G}^{(1)}$. In this case (assuming $T_\theta = 0$), we define the *curvature of the G-structure* at $f \in {\mathcal{F}_G}^{(1)}$ as an element $R_f = [R_\theta]_f$ of the Spencer cohomology group

$$H^{1,2}(\mathfrak{g}) := \frac{\ker \delta_2}{\mathrm{Image}\, \delta_1},$$

where $\delta_2 : \mathfrak{g} \otimes \Lambda^2 V^* \to V \otimes \Lambda^3 V^*$ and $\delta_1 : \mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 V^*$ are skew-symmetrization maps (see 5.7.3).

What to do if $H^{0,2}(\mathfrak{g}) \neq 0$ (or, more precisely, if $[T] \neq 0$). Here we must calculate how the torsion varies as one moves in the fiber, and choose some normalization of the torsion. This normalization of torsion will reduce the group of admissible changes of frame to a subgroup $H \subset G$, and we begin again, studying H-frames. Problems can occur if the G-action on the torsion is not nice, so we need a few definitions.

Consider the G-equivariant map

$$\begin{aligned} \tau : (\mathcal{F}_G)_x &\to H^{0,2}(\mathfrak{g}), \\ (x, u) &\mapsto [T_{(x,u)}] \end{aligned}$$

Definition 8.3.12. $H \subseteq G$ is said to be of *stabilizer type* if there exists $v \in H^{0,2}(\mathfrak{g})$ such that $H = \{g \in G \mid \mu(g)v = v\}$, where $\mu : G \to GL(H^{0,2}(\mathfrak{g}))$ is the induced representation.

Let H be of stabilizer type, and define $W_H := \{v \in H^{0,2}(\mathfrak{g}) \mid Hv = v\}$. Let $W_H^\bullet := \{v \in W_H \mid hv = v \Rightarrow h \in H\}$. A submanifold $S \subseteq W_H^\bullet$ is a *section* if for all $v \in S$, $Gv \cap S = \{v\}$ and S is transverse to Gv in $W_H^\bullet$ at v. A G-structure is of *type S* if $\tau(\mathcal{F}_G) \subseteq S$. In this case we can reduce to an H-structure, i.e., there is a one-to-one correspondence between G-structures of type S and H-structures $\mathcal{F}_H := \tau^{-1}(S)$.

Exercises 8.3.13:

1. Show that $W_H^\bullet$ is an open dense subset of W_H.
2. Show that $\mathcal{F}_H$ is smooth.

Example 8.3.14 (Almost complex manifolds). Let $\mathbb{R}^{2m}$ have an almost complex structure J and consider the real Lie algebra of derivations of J, $\mathfrak{g}(J) = \mathfrak{gl}_m(\mathbb{C}) \simeq V^{1,0*} \otimes V^{1,0}$.

Let M^{2m} be an almost complex manifold. Let $\{\omega^1, \dots, \omega^m, \overline{\omega}^1, \dots, \overline{\omega}^m\}$ be a coframing of M adapted to the splitting $V = V^{1,0} \oplus V^{0,1}$. Then the almost complex structure defined by J is integrable, and there exist local coordinates $(z^1, \dots, z^m, \overline{z}^1, \dots, \overline{z}^m)$ such that $(dz^1, \dots, dz^m, d\overline{z}^1, \dots, d\overline{z}^m)$ is a G-framing (i.e., the structure is flat), if and only if

$$d\omega^i \equiv 0 \mod \{\omega^1, \dots, \omega^m\},$$

i.e., if and only if there exist forms θ^i_j such that

$$d\omega^i = \theta^i_j \wedge \omega^j.$$

However, in general we have

$$d\omega^i = A^i_{jk}\omega^j \wedge \omega^k + B^i_{jk}\overline{\omega}^j \wedge \omega^k + C^i_{jk}\overline{\omega}^j \wedge \overline{\omega}^k$$

for some functions $A^i_{jk}, B^i_{jk}, C^i_{jk}$. Taking $\theta^i_k = A^i_{jk}\omega^j + B^i_{jk}\overline{\omega}^j$ eliminates as much torsion as permissible. Thus we see the non-absorbable torsion lives in $V^{1,0}\otimes\Lambda^2 V^{0,1*}$, so that $H^{0,1}(\mathfrak{g}) = V^{1,0}\otimes\Lambda^2 V^{0,1*}$.

8.4. Induced vector bundles and connections on induced bundles

We have defined and studied connections and differential invariants on $\mathcal{F}_G$. Our differential invariants were vector-valued functions on $\mathcal{F}_G$. It would be nicer to have differential invariants that were functions on M. We will now see how to push down our differential invariants to M, with the price of having *vector bundle* valued differential invariants. At the same time we see how a connection form on $\mathcal{F}_G$ induces a G-equivariant connection on TM and in fact on all vector bundles that arise naturally from the G-action on TM:

Induced vector bundles. Given a G-structure with bundle $\mathcal{F}_G$, or more generally any G-principal bundle over M, and a representation $\mu : G \to GL(W)$, we construct the *induced vector bundle* $E_\mu \to M$ by

$$E_\mu := (\mathcal{F}_G \times W)/\sim,$$

where the equivalence relation on $(u, w) \in \mathcal{F}_G \times W$ is $(u, w) \sim (ug, \mu(g^{-1})w)$.

Exercise 8.4.1: If $\rho : G \to GL(V)$ is the representation of G used in defining $\mathcal{F}_G$, then $TM = E_\rho$ and $T^*M = E_{\rho^*}$, where $\rho^* : G \to GL(V^*)$ denotes the dual representation.

For each of the Spencer cohomology spaces $H^{i,j}(\mathfrak{g})$ we define the induced vector bundles $\mathcal{H}^{i,j}(\mathfrak{g}) \to M$.

Remark 8.4.2. When $M = G/P$ is homogeneous, the vector bundle E constructed using a P-module W by the procedure above is also called homogeneous. A very special case is when G is a complex simple Lie group and $P = B$ is a *Borel* subgroup. (In the case of $SL(n, \mathbb{C})$, $SL(n, \mathbb{C})/B$ is the space of complete flags in $\mathbb{C}^n$ and in general G/B is a space of complete flags; see §3.3). Then, taking one-dimensional representations of B, one obtains holomorphic line bundles $L \to G/B$.

Let $\Gamma(L)$ denote the vector space of holomorphic sections of L. It is a finite dimensional vector space because G/B is a compact complex manifold. Moreover, it is naturally a G-module. The *Bott-Borel-Weil theorem* gives an explicit one-to-one correspondence between the irreducible finite dimensional G-modules and the vector spaces $\Gamma(L)$.

G-equivariant connections on induced vector bundles. The choice of a connection θ on $\mathcal{F}_G$ determines a *horizontal* distribution $\Delta \subset T\mathcal{F}_G$, i.e., a distribution complementary to the vertical tangent space $\mathcal{V}_p\mathcal{F}_G = \{w \in T_p\mathcal{F}_G \mid \pi_{*p}(w) = 0\}$, namely $\Delta := \ker\theta$.

Exercise 8.4.3: Show that, conversely, a G-equivariant horizontal distribution on $\mathcal{F}_G$ determines a connection form θ.

We have seen that a horizontal distribution on a vector bundle $E \to M$ determines a differential operator $\nabla : \Gamma(M,E) \to \Omega^1(M,E)$. A connection on $\mathcal{F}_G$ induces G-equivariant connections on all the E_μ by letting $\mathrm{pr} : \mathcal{F}_G \times W \to E_\mu$ denote the projection and defining the induced horizontal distribution $\Delta_{[(u,w)]} = \mathrm{pr}_*(\Delta_u \oplus 0)$. This gives a well-defined differential operator $\nabla = \nabla^\theta : \Gamma(M,E) \to \Omega^1(M,E)$.

We may also use the connection on $\mathcal{F}_G$ to arrive at a differential operator by following up on our first attempt to define a connection in §8.2. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{F}_G & \xrightarrow[(u,\tilde{s})]{} & \mathcal{F}_G \times W \\ \downarrow & & \downarrow \\ M & \xrightarrow[s]{} & E_\rho \end{array}$$

Given s, there is a corresponding map $(u,\tilde{s}) : \mathcal{F}_G \to \mathcal{F}_G \times W$ that makes the diagram commute. Then on $\mathcal{F}_G \times W$ define $\tilde{\nabla}\tilde{s} = d\tilde{s} + (\rho_* \circ \theta)s \in \Omega^1(\mathcal{F}_G \times W, W)$, where ρ_* is the induced Lie algebra representation.

Exercise 8.4.4: Show that $\tilde{\nabla}\tilde{s}$ descends to an element of $\Omega^1(M, E_\rho)$ and that this element is the same as ∇s defined by the induced horizontal distribution.

While $\tilde{\nabla}\tilde{s}$ is a vector-valued function on $\mathcal{F}_G$, to make things work out right when we push down, we added on an extra term to the naïve derivative. The correction term should come as no surprise, as in our example that worked, Example 8.2.3, we had $\nabla X = (dx^i + x^j s^*(\theta ij)) \otimes e_i$.

Proposition 8.4.5. *Let $\mathcal{F}_G \to M$ be a G-structure with connection form (θ^i_j), and ∇ the induced connection on $E = TM$. Let $s : M \to \mathcal{F}_G$ be any section, let e_i be the corresponding frame field, and let $X = x^j e_j \in \Gamma(TM)$.*

Then

$$\nabla X = (dx^i + x^j s^*(\theta^i_j)) \otimes e_i \in \Omega^1(M, TM).$$

The proof is along the lines of Example 8.2.3.

A G-equivariant connection on E induces connections on all tensor constructs of E, e.g., $S^k E$, $\Lambda^k E$, E^*, and $E \otimes E^*$, because it induces one on $\mathcal{F}_G$ (just reverse the above construction and throw away W). We will use the same symbol ∇ for all these connections.

Exercises 8.4.6:

1. Show that if $\nabla = \nabla^\theta$ is a connection operator on TM, then for all $X, Y \in \Gamma(TM)$, the maps $T : \Gamma(\Lambda^2 TM) \to \Gamma(TM)$ and $R : \Gamma(\Lambda^2 TM) \to \Gamma(\mathrm{End}(TM))$ defined by

i. $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$,

and

ii. $R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$

are actually tensors, i.e., at $x \in M$ they only depend on X_x, Y_x; thus $T(v,w)_x$, $R(v,w)_x$ are well-defined for $v, w \in T_x M$.

2. Show that T is zero iff the torsion of θ is zero; in fact, identify it with the torsion of θ. Show that $T \in \Gamma(M, \mathcal{H}^{0,2}(\mathfrak{g}))$.

3. Identify R with the curvature of θ. Show that $R \in \Gamma(M, \mathcal{H}^{1,2}(\mathfrak{g}))$.

4. Let ∇ be a connection on E . Show that ∇ is torsion-free iff $\nabla \, \mathrm{Id} = 0$, where $\mathrm{Id} \in \Gamma(E \otimes E^*)$ is the identity section.

5. Show that ∇ is torsion-free iff in any coordinate system the resulting Christoffel symbols Γ^i_{jk} satisfy $\Gamma^i_{jk} = \Gamma^i_{kj}\ \forall i, j, k$. Because of this, sometimes torsion-free connections are referred to as *symmetric* connections.

Induced forms. Given a G-structure $\mathcal{F}_G \to M$ and $\underline{\phi} \in \mathbb{T}(V)$ that is G-invariant, we obtain an induced form $\phi \in \overline{\Gamma(\mathbb{T}(TM))}$ as follows: Let $s : M \to \mathcal{F}_G$ be any section. Since $s(x)$ provides an isomorphism $T_x M \simeq V$, it induces an isomorphism $\mathbb{T}(V) \simeq \mathbb{T}(T_x(M))$ and we obtain a form $\phi = s^*(\underline{\phi})$ on M. Any other choice of section would only change our choice by a G-action, but G leaves $\underline{\phi}$ invariant so ϕ is well-defined.

Affine connections and projective structures. A connection ∇ on TM is sometimes called an *affine connection*. Given an affine connection, it is possible to define *geodesics* in M. Let $c : \mathbb{R} \to M$ be a parametrized curve, and let $c^*(\nabla) \in \Omega^1(\mathbb{R}, c^*(TM))$ denote the pullback of the connection from M. Define the geodesics on M to be the parametrized curves such that

$$\nabla_{c'} c' = 0,$$

where we have slightly abused notation, as we should have written $c^*(\nabla)_{\frac{\partial}{\partial t}} c^*(c') = 0$.

Define an equivalence relation on the space of torsion-free affine connections by saying ∇ is equivalent to $\tilde{\nabla}$ if they have the same geodesics. A choice of such an equivalence class on M is called a *projective structure* on M.

Let's look at this in local coordinates, as it will help in the understanding of path geometry, which we will study in §8.6. From now on, for simplicity, assume ∇ is torsion-free.

Write $c(t) = (x^1(t), \ldots, x^n(t))$ and $c'(t) = (x^i)'(t)\frac{\partial}{\partial x^i}$. We have

$$c^*(\nabla)c' = (x^i)''dt\otimes\frac{\partial}{\partial x^i} + (x^i)'(x^k)'\Gamma^j_{ik}dt\otimes\frac{\partial}{\partial x^j}.$$

So

$$c^*(\nabla)_{c'}c' = (x^i)'' + (x^j)'(x^k)'\Gamma^i_{jk}, \tag{8.6}$$

and we see that $c^*(\nabla)_{c'}c' = 0$ is a system of n second-order ODEs.

Now specialize to the case $\dim M = 2$ with coordinates (x, y).

Exercise 8.4.7: Eliminate t from (8.6) to obtain a single ODE:

$$\frac{d^2y}{dx^2} = \Gamma^1_{22}(\frac{dy}{dx})^3 + (2\Gamma^1_{12} - \Gamma^2_{22})(\frac{dy}{dx})^2(\Gamma^1_{11} - 2\Gamma^2_{12})(\frac{dy}{dx}) - \Gamma^2_{11}.$$

Proposition 8.4.8 (Cartan [**29**]). *A projective structure on a surface determines a second-order ordinary differential equation on the surface of the form*

$$\frac{d^2y}{dx^2} = A(\frac{dy}{dx})^3 + B(\frac{dy}{dx})^2C(\frac{dy}{dx}) + D$$

for some functions A, B, C, D on the surface. The solutions of this ODE are the geodesics of the projective structure.

In Proposition 8.6.26 we will see a converse to this.

8.5. Holonomy

In this section we assume M is connected and simply-connected; that is, we will only discuss the local theory.

Given a G-structure $\mathcal{F}_G \to M$, we ask: Is there some proper subgroup $H \subset G$ that induces the "same geometry" as G? (Our question will be made precise below.) For example, with a flat G-structure it is the subgroup $\mathrm{Id} \subset G$ that determines the geometry.

Our first task is to make our question precise. One idea is that the curvature $\Theta \in \Omega^2(M, \rho(\mathfrak{g}))$ and torsion $\tau \in \Gamma(M, \mathcal{H}^{0,2}(\mathfrak{g}))$ should take values in a smaller Lie algebra, e.g., $\Theta \in \Omega^2(M, \rho(\mathfrak{h}))$ for some $\mathfrak{h} \subset \mathfrak{g}$. We will soon see that this is the correct idea (see the Ambrose-Singer Theorem below), but first, in preparation, we have to develop some terminology.

Our notion will be provided by the *holonomy* of a connection. Recall that in a vector space, we may identify the tangent spaces at all points and thus move vectors around from one tangent space to another. We can't quite do this on a manifold, even one equipped with a connection, but part of this can be salvaged: we can move vectors along curves.

Parallel transport. Let $\pi : \mathcal{F}_G \to M$ be a G-structure with torsion-free connection θ. Let $\alpha : [a, b] \to M$ be a smooth curve, let $x_0 = \alpha(a)$ and let $u_0 \in \pi^{-1}(x_0)$.

Exercise 8.5.1: Show that there exists a unique curve $\tilde{\alpha} : [a, b] \to \mathcal{F}_G$ such that $\tilde{\alpha}(a) = u_0$, $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}^*(\theta) = 0$. ⊚

We call $\tilde{\alpha}$ the *horizontal lift of* α through u_0, and curves that are horizontal lifts of some curve in the base will be referred to as *horizontal curves.*

Exercise 8.5.2: Show that if $\tilde{\alpha}(t)$ is a horizontal curve in $\mathcal{F}_G$, then $\beta(t) = \tilde{\alpha}(t) \cdot g$ is also horizontal for any $g \in G$. ⊚

For those of you wondering about the origin of the terminology "connection", parallel translation allows us to "connect" tangent spaces at different points of M, which evidently was the motivation for the terminology. To see this, consider the linear map

$$\tau_\alpha : T_{\alpha(a)}M \to T_{\alpha(b)}M$$

defined by $\tau_\alpha = \tilde{\alpha}(b)^{-1} \circ \tilde{\alpha}(a) : T_{\alpha(a)}M \to V \to T_{\alpha(b)}M$. This τ_α is called *parallel translation along* α *with respect to the connection* θ. Note that it is independent of the choice of $u \in \pi^{-1}(x_0)$, because if $u' \in \pi^{-1}(x_0)$ and τ'_α is the corresponding linear map, then $u' = g^{-1} \circ u$ for some $g \in G$ and $\tilde{\alpha}'(t) = \tilde{\alpha}(t) \cdot g$ for $t \in [a, b]$; see Figure 3.

Given $u \in \mathcal{F}_G$, let

$$\mathcal{P}^\theta_u = \{u' \in \mathcal{F}_G \mid \alpha : [a, b] \to M,\ \tilde{\alpha}(a) = u, \tilde{\alpha}(b) = u'\}.$$

In other words, $\mathcal{P}^\theta_u$ is the set of points in the bundle which can be reached from u by following horizontal lifts of curves in M. This is called the *holonomy bundle* of θ through u, and its structure group is called the holonomy group of θ:

Definition 8.5.3. For $u \in \mathcal{F}_G$, let $\mathrm{Hol}^\theta_u = \{h \in G | u \cdot h \in \mathcal{P}^\theta_u\}$, the *holonomy group* of θ relative to u.

Proposition 8.5.4. *The holonomy groups are connected subgroups of G. If u, u' are connected by a horizontal curve in $\mathcal{F}_G$, then $\mathrm{Hol}^\theta_u = \mathrm{Hol}^\theta_{u'}$. If u, u' are in the same fiber of $\mathcal{F}_G$, then Hol^θ_u and $\mathrm{Hol}^\theta_{u'}$ are conjugate in G.*

Exercise 8.5.5: Prove the proposition.

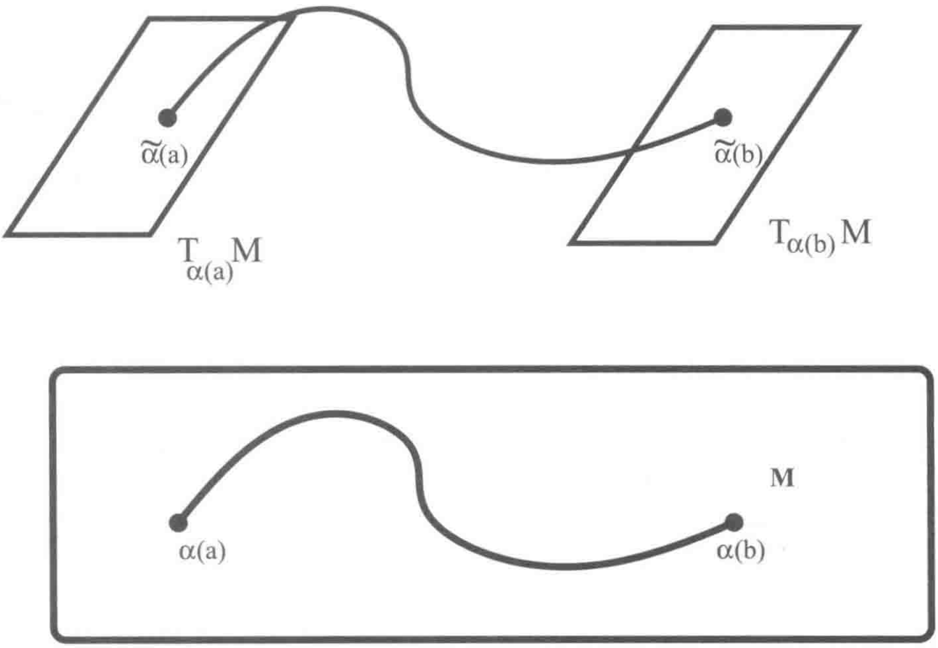


Figure 3. Parallel translation

We may define the holonomy group independent of a reference coframe u, if we are willing to forego an explicit embedding of the group into the matrix group G:

Exercise 8.5.6: Suppose $\alpha(a) = \alpha(b) = x$; then we obtain a map $\tau_\alpha : T_xM \to T_xM$. We may define

$$\mathrm{Hol}^\theta_x = \{\tau_\alpha \mid \alpha : [a,b] \to M,\ \alpha(a) = \alpha(b) = x\} \subseteq GL(T_xM)$$

Let $u \in \pi^{-1}(x) \subset \mathcal{F}_G$. Then $\mathrm{Hol}^\theta_x \simeq \mathrm{Hol}^\theta_u$ under the isomorphism between $GL(T_xM)$ and G induced by $u : T_xM \to V$.

Since each Hol^θ_x is isomorphic to a subgroup of G (which is well-defined up to conjugation), we will drop reference to x in the notation.

We may understand Hol^θ as measuring the failure of parallel translation along closed curves to take a vector to itself.

We now explore some relations of holonomy with infinitesimal geometry:

Exercise 8.5.7: Let $E = E_\rho \to M$ be an induced vector bundle and let $s : M \to E$ be a section. Let $X \in T_xM$ and let x_t be a curve with $x_0 = x$ and $x_0' = X$. Let $\tau^t = \tau_{x_t|_{[0,t]}}$ be the parallel translation along x_t with respect to the connection ∇ on E. Show that

$$\nabla_X s = \lim_{t \to 0} \frac{1}{t}[\tau^t(s(x_t)) - s(x)].$$

Given a G-structure, with torsion-free connection θ, our question posed in the introduction to this section may be stated more precisely as: When is Hol^θ is a proper subgroup of G?

Proposition 8.5.8. *Let V be a G-module and let $W \subseteq V^{\otimes k} \otimes V^{*\otimes l}$ be a G-submodule. Let $E_\rho \to M$ be the vector bundle induced by $\rho : G \to GL(W)$. If there exists $\phi \in \Gamma(E_\rho)$ such that $\nabla\phi = 0$, then*

$$\mathrm{Hol}^\theta_u \subseteq G_\phi := \{g \in G \mid \rho(g)\phi = \phi\}.$$

We will say that such a ϕ is *G-parallel.*

Proof. The result is local, so fix a local G-framing $e_1, \dots, e_n$ with dual coframing $\omega^1, \dots, \omega^n$ giving a section $s : M \to \mathcal{F}_G$ through u. We may write

$$\phi = p^{i_1,\dots,i_k}_{j_1,\dots,j_l} \omega^{j_1} \otimes \dots \otimes \omega^{j_l} \otimes e_{i_1} \otimes \dots \otimes e_{i_k}$$

Then

$$\begin{aligned}\nabla\phi &= dp^{i_1,\dots,i_k}_{j_1,\dots,j_l} \omega^{j_1} \otimes \dots \otimes \omega^{j_l} \otimes e_{i_1} \otimes \dots \otimes e_{i_k} \\ &+ p^{i_1,\dots,i_{s-1},m,i_{s+1},\dots,i_k}_{j_1,\dots,j_l} s^*(\theta^{i_s}_m) \otimes \omega^{j_1} \otimes \dots \otimes \omega^{j_l} \otimes e_{i_1} \otimes \dots \otimes e_{i_k} \\ &+ p^{i_1,\dots,i_k}_{j_1,\dots,j_{s-1},m,j_{s+1},\dots,j_l} {}^t s^*(\theta^{j_s}_m) \otimes \omega^{j_1} \otimes \dots \otimes \omega^{j_l} \otimes e_{i_1} \otimes \dots \otimes e_{i_k}.\end{aligned}$$

Now let $\alpha : [a, b] \to M$ be a curve and $\tilde{\alpha}$ its horizontal lift. By definition, $\tilde{\alpha}^*(\theta) = 0$, so if ϕ is G-parallel, then $dp^{i_1,\dots,i_k}_{j_1,\dots,j_l}(\alpha') = 0$, i.e., the functions $p^{i_1,\dots,i_k}_{j_1,\dots,j_l}$ are constant on lifts of horizontal curves; so $\tau^*(\phi) = \phi$, and thus $\mathrm{Hol}^\theta_u \subseteq G_\phi$. □

Exercise 8.5.9: Show conversely that $\mathrm{Hol}^\theta_u \subseteq G_\phi$ implies $\nabla\phi = 0$.

Corollary 8.5.10. *Flat G-structures have* Hol = Id.

Example 8.5.11. Let (M^2, g) be a surface with a Riemannian metric. If the structure is not flat, then $\mathrm{Hol}^\theta \simeq SO(2)$, because there are no Lie groups between the identity and $SO(2)$.

Corollary 8.5.12. *If Hol^θ is reductive and preserves a subspace $L \subset T_xM$, then $\mathrm{Hol}^\theta \subseteq GL(L) \times GL(L^c)$. In particular, if G occurs as an irreducible holonomy group (i.e., $G \neq G_1 \times G_2$) of a Riemannian metric, it must act transitively on the sphere.*

Exercise 8.5.13: Prove 8.5.10 and 8.5.12.

In the case of Riemannian geometry, something stronger is true:

Theorem 8.5.14 (de Rham). *Let (M, g) be Riemannian, complete, and simply-connected. Say $T_xM = L_1 \oplus L_2$ with each L_j preserved by Hol^θ. Then (M, g) is isometric to a product $(M_1 \times M_2, g_1 \oplus g_2)$ with $T_xM_i = L_i$.*

For the proof, see ([**94**], Chapter IV).

We now state and prove the Ambrose-Singer Theorem relating holonomy and curvature:

Theorem 8.5.15 (Ambrose-Singer). *Let $\mathcal{F}_G \to M$ be a G-structure with connection θ. For $u \in \mathcal{F}_G$, let*

$$W := \{\Theta(v, w) \mid v, w \in T_u\mathcal{F}_G\}.$$

Then $W = \mathfrak{h}$, the Lie algebra of Hol_u^θ.

Proof (adapted from [**94**]). First, note that $W \subseteq \mathfrak{g}$ because, since θ is $\mathfrak{g}$-valued, so is $\Theta = d\theta + \theta \wedge \theta$. Moreover, the connection restricted to $\mathcal{P}_u$ is $\mathfrak{h}$-valued. Hence, it suffices to assume that $\mathcal{F}_G = \mathcal{P}_u$ and $\mathfrak{g} = \mathfrak{h}$.

Next, we show that W is an ideal in $\mathfrak{g}$. In general, $U \subseteq \mathfrak{g}$ is an ideal (i.e., $[U, \mathfrak{g}] \subseteq U$) iff $\mathrm{Ad}_G(U) \subseteq U$. Now, let $w = \Theta(v_1, v_2)$. Then

$$\mathrm{Ad}_g(w) = gwg^{-1} = \Theta(R_{g^{-1}}v_1, R_{g^{-1}}v_2) \in W.$$

Next, fix a splitting $\mathfrak{g} = W \oplus W^c$ and write $\theta = \theta' + \theta''$, where θ' is W-valued and θ'' is W^c-valued. Let I be the Pfaffian system on $\mathcal{F}_G$ spanned by the components of θ''; note that I is dual to a distribution of $(n+k)$-planes, where $k = \dim W$. Then

$$\Theta = (d\theta' + \theta' \wedge \theta') + (d\theta'' + \theta'' \wedge \theta'') + (\theta' \wedge \theta'' + \theta'' \wedge \theta').$$

Projecting both sides into W^c shows that $0 \equiv d\theta'' \mod I$, and thus I is Frobenius.

Let $Q \subseteq \mathcal{P}_u$ be the maximal integral manifold of I through u. Since any horizontal curve in $\mathcal{F}_G$ is an integral of θ'', then $Q = \mathcal{P}_u = \mathcal{F}_G$. Hence $k = \dim W = \dim Q - n = \dim \mathcal{F}_G - n = \dim \mathfrak{g}$ and $W = \mathfrak{g}$. □

We will continue to discuss holonomy, but first we take the following *Detour into symmetric spaces.* On $M = \mathbb{E}^n$ and $M = S^n$ one can define a "reflection" through each point, that is, for $x \in M$ an isometry $\sigma_x : M \to M$ such that $\sigma_x^2 = \mathrm{Id}$ and x is an isolated point of σ. A Riemannian manifold is defined to be *symmetric* if it has such a reflection isometry through each point. (More generally, it is *locally symmetric* if there is a reflection isometry defined on a neighborhood around each point.) Lie groups G equipped with a bi-invariant metric are symmetric—one takes σ_e to be the map $g \mapsto g^{-1}$ and then translates this mapping around the group by the adjoint action, i.e., $\sigma_g(h) = gh^{-1}g^{-1}$.

In fact, a connected symmetric space is necessarily homogeneous. The key point is that the isometry group G of a symmetric space acts transitively on it, so it is a homogeneous space of G. However, as the following remark

shows, not all homogeneous spaces with G-invariant Riemannian metrics are symmetric.

Remark 8.5.16 (Symmetric Lie algebras). Suppose a connected Riemannian manifold $M = G/H$ is symmetric. Without loss of generality, we may assume that G is the connected component of the isometry group of M. Let $x = eH$ be the identity coset or "origin" of M; then H fixes x and acts on T_xM by infinitesimal isometries.

Let σ_0 be reflection at x. Define an automorphism of G by $\rho(g) = \sigma_0 \circ g \circ \sigma_0^{-1}$. Since ρ is an automorphism, its fixed point set is a subgroup $G' \subset G$. If $g \in G'$, then g commutes with σ_0, so $g(x)$ is a fixed point of σ_0. Thus, the identity component of G' is H.

Hence, within $\mathfrak{g} = T_eG$, the $+1$-eigenspace of ρ_* is $\mathfrak{h}$. Let V be the -1-eigenspace, which projects down to T_xM, and give V the inner product from T_xM. Because ρ_* is an infinitesimal automorphism of $\mathfrak{g}$, it distributes across Lie brackets, i.e. $\rho_*[v, w] = [\rho_* v, \rho_* w]$. So, Lie brackets of the form $[\mathfrak{h}, V]$ must lie in V. Since the adjoint action of H on T_eG covers the action of H on T_xM, then the adjoint representation of $\mathfrak{h}$ lies in $\mathfrak{so}(V)$.

Exercise 8.5.17: Show that $[V, V] \subseteq \mathfrak{h}$.

These considerations reduce the classification of Riemannian symmetric spaces to the classification of *orthogonal involutive Lie algebras* [**77**], i.e. Lie algebras with an involutive automorphism, such that the adjoint action of the $+1$-eigenspace on the -1-eigenspace respects an inner product. See [**77**] for the classification.

Here is a simple criterion for a symmetric metric:

Proposition 8.5.18. *Let (M, g) be a Riemannian manifold with Levi-Civita connection θ and curvature Θ. If $\nabla(\Theta) = 0$, then (M, g) is locally symmetric.*

Proof. Let $x \in M$ and fix an orthonormal coframe u at x. On the holonomy bundle $\mathcal{P}_u$, let $\Theta^i_j = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l$. Our assumption implies that R^i_{jkl} is constant on $\mathcal{P}_u$. Let $\mathfrak{h}$ be the Lie algebra of Hol^θ_u. Let $\mathfrak{g} = \mathfrak{h} \oplus V$ as a vector space, with the Lie brackets of the form $[\mathfrak{h}, V]$ defined by $[h, v] = hv \in V$ (via matrix multiplication) and of the form $[V, V]$ defined by $[e_k, e_l] = R^i_{jkl} e_i \otimes e^j \in \mathfrak{h}$.

Exercise 8.5.19: Show that $\mathfrak{g}$ is a Lie algebra, i.e., the structure constants satisfy the Jacobi identity.

Let G be the corresponding simply-connected Lie group and let $H = \exp(\mathfrak{h})$. We define an involutive automorphism σ on G by letting σ_{e*} be $+1$

on $\mathfrak{h}$ and -1 on V, and then exponentiate and translate around G as above. The adjoint action preserves the standard inner product on V, and so G/H is a symmetric space.

On $\mathcal{P}_u$, the structure equations

$$d\omega^i = -\theta^i_j \wedge \omega^j,$$
$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \tfrac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l$$

imply that the $(n+1)\times(n+1)$ matrix ψ, with entries ψ^α_β defined by $\psi^0_\beta = 0$, $\psi^k_0 = \omega^k$ and $\psi^i_j = \theta^i_j$, satisfies the Maurer-Cartan equation. It follows from Theorem 1.6.10 that near u there is a local diffeomorphism $\varphi : \mathcal{P} \to G$ with $\varphi^*(g^{-1}dg) = \psi$. Consequently, φ carries the fibers to cosets of H, and φ covers a local isometry between a neighborhood of $x \in M$ and a neighborhood of the identity in G/H. $\square$

Special holonomy. Symmetric spaces with irreducible holonomy have been classified. As mentioned in 8.5.16, their classification amounts to classifying automorphisms of Lie algebras. So now we have a refined question: which groups $G \subseteq SO(n)$ arise as the holonomy of a non-symmetric metric?

We need to have curvature Θ such that $\nabla\Theta \neq 0$; in particular, the space where $\nabla\Theta$ lives cannot be empty.

Exercise 8.5.20: If $\mathrm{Hol}^\theta \subseteq \mathfrak{g}$, show that $\nabla\Theta_x \in H^{2,2}(\mathfrak{g})$.

In summary, we have two necessary conditions for G to be the holonomy of a non-symmetric metric:

1. There does not exist $\mathfrak{g}' \subsetneq \mathfrak{g}$ with $H^{1,2}(\mathfrak{g}) = H^{1,2}(\mathfrak{g}')$ (because then the holonomy would have to be contained in $\mathfrak{g}'$ by the Ambrose-Singer Theorem).

2. $H^{2,2}(\mathfrak{g}) \neq 0$.

Theorem 8.5.21 (Berger, Alexeevski, Brown-Gray). *The Lie algebras $\mathfrak{g} \subseteq \mathfrak{so}(n)$ satisfying 1 and 2 are*

1. $\mathfrak{so}(n)$,
2. $\mathfrak{u}(n/2)$,
3. $\mathfrak{su}(n/2)$,
4. $\mathfrak{sp}(1) + \mathfrak{sp}(n/4)$,
5. $\mathfrak{sp}(n/4)$,
6. $\mathfrak{g}_2 \subset \mathfrak{so}(7)$,
7. $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$.

Which of these actually occur outside of symmetric spaces? In fact, they all do.

Getting $SO(n)$ is easy: take the standard metric on the sphere and perturb it a little. The holonomy group will not shrink, therefore it stays $SO(n)$. The same method applied naïvely won't work for the other groups, as the holonomy could jump up to $SO(n)$ under a deformation. Non-symmetric Kähler metrics exist; just take any non-homogeneous subvariety of complex projective space equipped with its Fubini-Study metric.

Cases 3,4,5 can be handled locally using a refined deformation theory; see [**11**].

Most recently resolved were cases 6 and 7. They were first resolved locally by Bryant [**17**], and then later complete [**24**] and compact [**86, 87**] examples were found. We outline Bryant's existence proof in the following subsection. It is still not known whether or not compact examples exist for case 4.

Exterior differential systems for torsion-free G-structures.

A tautological system. We set up an EDS whose integral manifolds correspond to torsion-free (1-flat) G-structures $s : M \to (\mathcal{F}_{\mathsf{GL}}/G)$. Let $\tilde{\pi}_G : \mathcal{F}_{\mathsf{GL}}/G \to M$ denote the projection and let $\mathcal{H} = \mathcal{H}^{0,2}(\mathfrak{g}) = \mathcal{F}_{\mathsf{GL}} \times_G H^{0,2}(\mathfrak{g})$.

To calculate the torsion, we don't need a G-structure, but merely its 1-jet. Let $\mathcal{H}'$ denote the pullback of $\mathcal{H}$ to $J^1(M, \mathcal{F}_{\mathsf{GL}}/G)$. We have a natural map

$$\tau : J^1(M, \mathcal{F}_{\mathsf{GL}}/G) \to \mathcal{H}'$$

such that $\tau(j^1(s)) = [T_s]$. Let $\Sigma = \tau^{-1}(0) \subset J^1(M, \mathcal{F}_{\mathsf{GL}}/G)$, and let (I, J) denote the pullback of the contact system on $J^1(M, \mathcal{F}_{\mathsf{GL}}/G)$ to Σ. Then the integral manifolds of the system (I, J) are exactly the lifts of 1-flat G-structures on M to $J^1(M, \mathcal{F}_{\mathsf{GL}}/G)$.

Similarly, define a linear Pfaffian system for 2-flat G-structures by intersecting the prolongation of (I, J) with $\mathbf{r}^{-1}(0)$, where $\mathbf{r} : J^2(M, \mathcal{F}_{\mathsf{GL}}/G) \to \mathcal{H}^{1,2}(\mathfrak{g})$ is the curvature mapping.

Torsion and invariant forms. Let $s : M \to \mathcal{F}_{\mathsf{GL}}/G$ be a G-structure and let $\underline{\phi} \in \Lambda^k(V^*)$ be G-invariant. As seen in §8.4, it induces a form $\phi \in \Omega^k(M)$. We will show that a necessary condition for s to be torsion-free is that $d\phi = 0$. We already know that $\nabla\phi = 0$ by Exercise 8.5.9.

First observe that if s is flat, then $d\phi = 0$. This is because, if s is flat, take local coordinates x^i such that $s = (dx^i)$. Write $\phi = a_I dx^I$ (using multi-index notation). We have $0 = \nabla\phi = (da_I + a_I\rho^*(\theta)) \otimes dx^I$, but $\rho^*(\theta) = 0$.

Now if a G-structure is 1-flat, the above is still true to first order, since by Exercise 8.3.6, for all $x \in M$, there exist local coordinates x^i centered at x, such that *at* x, $s_x = (dx^1, \dots, dx^n)$ and $\rho_x^*(\theta) = 0$. Thus $d\phi_x = 0$ for all $x \in M$.

In terms of representation theory, the above argument implies that if $\phi \in \Lambda^k V^*$ is G-invariant, then a component of $\Lambda^{k+1}V^*$ must occur in the G-module in $H^{0,2}(\mathfrak{g})$, as $d\phi_x \in \Lambda^{k+1}T_xM$ must be zero if the connection is torsion-free, so $d\phi_x$ represents a component of the torsion.

Example 8.5.22. Recall from Appendix A that

$$\mathfrak{sp}(V) \simeq S^2V^*,$$
$$\delta(S^2V^* \otimes V^*) = S_{21}V^*$$

Since $V^* \otimes \Lambda^2 V^* \simeq \Lambda^3 V^* \oplus S_{21}V^*$, we see that $H^{0,2}(\mathfrak{sp}(V)) \simeq \Lambda^3 V^*$ as $\mathfrak{sp}(V)$-modules. Since ϕ is a generic form, its exterior derivative can hit both irreducible components of $\Lambda^3 T_x^* M$, so it accounts for all of the torsion, giving a fancy proof of Exercise 8.1.18.

Manifolds with G_2 holonomy. Recall that $G_2(\phi) \subset GL(7,\mathbb{R})$ is the group preserving a generic positive $\phi \in \Lambda^3\mathbb{R}^7$ (see §A.5). Write $V = \mathbb{R}^7$. Since $G_2 \subset SO(7)$, we may identify V with V^*.

By Exercise 8.3.6.1, $H^{0,2}(\mathfrak{g}_2) \simeq (\mathfrak{so}(7)/\mathfrak{g}_2) \otimes V = (\Lambda^2 V/\mathfrak{g}_2) \otimes V = V \otimes V$ as a $\mathfrak{g}_2$-module. A modest computation or a quick click on the computer program LiE [**115**] shows that as a $\mathfrak{g}_2$-module, $V \otimes V = V_{20} \oplus \mathfrak{g}_2 \oplus V \oplus \mathbb{R} = V_{20} \oplus V_{01} \oplus V_{10} \oplus V_{00}$. (Here we use the convention that V_{ij} is of highest weight $i\omega_1 + j\omega_2$; see [**52, 15**].)

Moreover, $\Lambda^4 V = V_{2,0} \oplus V \oplus \mathbb{R}$, so the exterior derivative of ϕ can only fill three components of $H^{0,2}$. However, there is another G_2-invariant differential form, coming from $*\phi \in \Lambda^4 V$. We have $\Lambda^5 V \simeq \Lambda^2 V = V \oplus \mathfrak{g}$. One needs to check that the exterior derivative or $*\phi$ can actually hit the $\mathfrak{g}$ factor in $\Lambda^4 T_x^* M$. (Aside: once $(d\phi)_x$ is known, the component of $(d(*\phi))_x$ in the V factor in $\Lambda^5 V$ is determined.) After showing this, one obtains an EDS for 7-manifolds with torsion-free connections whose holonomy is at most G_2, as follows:

Let Σ denote the open subset of $\Lambda^3 T^* M$ consisting of nondegenerate forms. We obtain an exterior differential system for closed and co-closed 3-forms generated in degrees 4 and 5. Bryant shows this system is involutive. What remains to show is that one, in fact most, integral manifolds have holonomy equal to G_2, which he does by examining the exterior differential system for 2-jets of torsion-free G_2-structures and showing that its space of local integral manifolds is strictly smaller. The proof for Spin(7) is similar.

After Bryant's local existence results, Bryant and Salamon found complete examples [**24**] and soon afterwards D. Joyce [**86, 87**] found compact examples for both G_2 and $Spin(7)$. Joyce's constructions use methods coming from algebraic geometry (Kummer construction, deformation theory) and analysis instead of EDS, although they are inspired by Bryant's method.

As Bryant showed, in the G_2 case one needs a compact 7-manifold equipped with a 3-form that is closed and co-closed. Joyce begins by constructing (with the aid of deformation theory) compact 7-manifolds equipped with a 3-form ϕ that is closed and such that $d(*\phi)$ is small. He then uses analytical tools to show that ϕ can be perturbed into a form that is both closed and co-closed.

8.6. Extended example: Path geometry

The reader may also find discussions of path geometry in [**128**] and in the appendix to [**23**].

Classical formulation of the problem. Given an ODE, it would be useful to know if it is a familiar ODE in disguise. Even better would be to have a classification of ODE's up to some notion of 'equivalence'. We already have seen one notion of equivalence for first-order ODE's in the case of 3-webs in $\mathbb{R}^2$. The G-structure we now study will enable us to classify second-order ODE in $\mathbb{R}^2$ up to the following notion of equivalence:

Definition 8.6.1. We will say that two second-order ordinary differential equations in $\mathbb{R}^2$

$$y'' = F(x, y, y'), \qquad y'' = G(x, y, y')$$

are *equivalent under point transformations* if there exists a diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ taking solutions of one equation to solutions of the other.

We state our question as follows:

Problem 8.6.2. Find invariants of second-order ODE that classify them up to point transformations. Interpret the invariants in terms of useful properties, e.g., which second-order ODE's admit first integrals (and are therefore solvable by quadrature)?

Fancy formulation of the problem. Let X be a surface. A *path geometry* on X is a set of smooth curves $\mathcal{C}$ on X such that, for all $x \in X$ and for all directions $l \in \mathbb{P}T_xX$, there is a unique $\gamma \in \mathcal{C}$ such that $x \in \gamma$ and $T_x\gamma = l$.

Problem 8.6.3. Locally classify path geometries in the plane by finding differential invariants of path geometries. Interpret the invariants in terms of familiar path geometries, such as geodesics of a projective structure on TX or ODE in the plane.

We will see below that every path geometry is locally equivalent to the path geometry induced by the solutions to a second-order ODE.

Example 8.6.4. Let X have a Riemannian or more generally a Finsler metric, and take the paths to be the geodesics.

Example 8.6.5. Let X have a projective structure and take the paths to be the geodesics.

Example 8.6.6. Let the paths be circles of radius one in $X = \mathbb{R}^2$.

Definition 8.6.7. We will call a set of paths in X *flat* if for any $x \in X$ there is an open set $U \subset X$, $x \in U$, and a local diffeomorphism from U to an open set in $\mathbb{R}^2$ that carries the paths in U to straight line segments. In other words, the flat case locally corresponds to solutions of the ODE $y'' = 0$.

Problem 8.6.8. Which Riemannian metrics on a surface induce a flat set of paths?

It is clear that the Euclidean metric on $\mathbb{R}^2$ induces the flat set of paths, but are there others?

Proposition 8.6.9. *S^2 with its standard metric induces the flat set of paths.*

Proof. Since $S^2 \subset \mathbb{R}^3$ is homogeneous, it suffices to define a local diffeomorphism from a neighborhood $U \subset S^2$ of the north pole N to $\mathbb{R}^2$ which carries great circles to straight lines. To do this, we use a stereographic projection mapping a point P to the point where the line OP cuts the plane $T_N S^2$ (where O is the origin). □

Exercise 8.6.10: Prove that the set of paths of geodesics on hyperbolic space H^2 is flat. ◎

Remark 8.6.11. A path geometry is a set of second-order data on X, i.e., it does not correspond to a foliation or other kind of distribution on X. In order to study it using a first-order G-structure, we must work on a space such that the path data becomes first order. The problem of classifying path geometries up to equivalence can be viewed as an example of a generalization of the notion of a G-structure called a *G/H-structure of order two*, due also to Cartan. See **[139]** where it is a special case of what is called a *Cartan geometry*.

A path is determined by a point $p \in X$ and a tangent direction $[\mathsf{v}] \in \mathbb{P}(T_pX)$. Let $\mathcal{P}$ be the projectivized tangent bundle $\mathbb{P}(TX)$ over X. Later in this section, we will set up the problem of equivalence of path geometries as a first-order G-structure problem on $\mathcal{P}$. For now, we make several observations:

We lift each path in X to a curve in $\mathcal{P}$ by taking its projectivized 1-jet. This gives a 2-parameter family of curves in $\mathcal{P}$ which foliates $\mathcal{P}$. (To see the two parameters, one has a one-parameter family of paths through each point of X plus two dimensions worth of points on X minus one dimension

for counting a path for each point it passes through.) Call the leaves of this foliation L_1. We will also use L_1 to refer to the distribution of tangent spaces to the leaves.

Notice that the fibration

$$\begin{tikzcd} S^1 = \mathbb{RP}^1 \arrow[r] & \mathcal{P} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

gives another foliation of $\mathcal{P}$ by the fibers of π. Call the leaves of this foliation L_2.

Suppose that the quotient of $\mathcal{P}$ by the leaves of L_1 is a smooth surface Z. Then Z is the space of paths in X, and we have the following *double fibration*:

$$\begin{tikzcd} & \mathcal{P} \arrow[dl] \arrow[dr] & \\ \mathcal{P}/L_1 = Z & & X = \mathcal{P}/L_2 \end{tikzcd}$$

(If the path geometry results from a second-order ODE in the plane, we may think of Z as the space of initial conditions for (or solutions to) the ODE.) While a point in Z corresponds to a path in X, a point $x \in X$ also gives a curve in Z (the 1-parameter family of paths passing through x). The resulting set of paths in Z is called the *dual path geometry*.

The space $\mathcal{P}$ may be thought of as an *incidence correspondence*

$$\begin{aligned} \mathcal{P} &= \{(z,x) \mid x \in \text{some path in } X \text{ corresponding to } z\} \\ &= \{(z,x) \mid z \in \text{some path in } Z \text{ corresponding to } x\}. \end{aligned}$$

This notion is motivated by the following important example, which will turn out to be our "flat" model:

Example 8.6.12. Let $X = \mathbb{RP}^2$ have the standard set of straight-line paths. Then $Z = \mathbb{P}^{2*}$, the set of lines in $\mathbb{P}^2$, and $\mathcal{P}$ is the flag manifold $\mathbb{F}_{12} := \{(x,H) | x \in H\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$. The paths in $\mathbb{P}^{2*}$ are the linear $\mathbb{P}^{1*}$'s, so the picture is completely symmetric.

The map from $SL(3,\mathbb{R})$ to pairs (x,H) is given in bases by $(e_1,e_2,e_3) \mapsto (\{e_3\},\{e_2,e_3\})$.

Remark 8.6.13. In dimensions greater than two, the space of paths is of higher dimension than the original space. For example, if $X = \mathbb{P}^n$, then $Z = G(\mathbb{P}^1,\mathbb{P}^n) = G(2,n+1)$ has dimension $2(n-1)$. Here $\mathcal{P}$ is the space of flags, $\hat{x} \subset \hat{l} \subset V$, and there is no naturally induced path geometry on Z.

On the other hand, one can recover the duality by restricting to a subset of the space of all paths on X, i.e., restricting to certain submanifolds of Z of the same dimension as X. See [**71**], [**9**] for this story and its relevance for the Radon transform.

We define a *path geometry* on a 3-fold M to be the following data: two line bundles $L_1, L_2 \subset TM$ such that

(a) $(L_1)_x \cap (L_2)_x = 0$, $\forall x \in M$, and

(b) the 2-plane distribution $L_1 + L_2$ is nowhere integrable.

Example 8.6.14. Let X be a surface with a Riemannian metric; then we may take $M = \mathcal{F}_{on}(X)$ and $L_1 = \{\omega^2, \omega^2_1\}^\perp$, $L_2 = \{\omega^1, \omega^2\}^\perp$.

Exercises 8.6.15:

1. Show that the L_1-curves are liftings of geodesics in X.
2. Show that, if we lift the geodesics to define a path geometry on $M' = \mathbb{P}(TX)$ in the usual way, then the two path geometries are locally diffeomorphic via a covering $p : M \to M'$ that commutes with the fibrations.

We will use the following as a running example:

Example 8.6.16. For an ODE $y'' = F(x, y, y')$, let (x, y, p) be coordinates on $J^1(\mathbb{R}, \mathbb{R}) = M$. Let

$$L_1 = \{dp - Fdx, dy - pdx\}^\perp, \tag{8.7}$$

$$L_2 = \{dx, dy\}^\perp. \tag{8.8}$$

Then

$$\{dy - pdx\}^\perp = L_1 + L_2 \tag{8.9}$$

is a contact distribution.

Conversely, for any path geometry on a manifold M, we can find locally defined functions x, y satisfying (8.8) by the Flowbox Theorem 1.2.2. Since $(L_1 + L_2)^\perp \subset L_2^\perp$, there must be functions p, q such that $(L_1 + L_2)^\perp = \{qdy - pdx\}$, and since we are working locally, perhaps after switching the roles of x and y, we may assume $q \equiv 1$, implying (8.9). The non-integrability condition on $L_1 + L_2$ implies that x, y, p are local coordinates by Pfaff's Theorem 1.9.17. Then since L_1 is transverse to L_2, $L_1{}^\perp$ must be spanned by $dy - pdx$ plus something of the form $-Fdx + \beta dy + \gamma dp$, and we can assume $\beta \equiv 0$ by adding on a multiple of $dy - pdx$. Furthermore, since we know $\gamma \neq 0$, we may assume $\gamma \equiv 1$. Thus *any path geometry is locally equivalent to one arising from a second-order ODE.*

Setting up the G-structure. Let $G \subset GL(3,\mathbb{R})$ be the subgroup preserving the spans $\{e_1\}$ and $\{e_3\}$:

$$G = \left\{ \begin{pmatrix} a & e & 0 \\ 0 & b & 0 \\ 0 & f & c \end{pmatrix} \middle| \, abc \neq 0 \right\}.$$

Let M^3 be equipped with a path geometry, and define a G-structure on M by

$$B = \mathcal{F}_G = \{u \in \mathcal{F}(M) \mid u(L_1) = \{e_1\},\ u(L_2) = \{e_3\}\}.$$

Let $\omega = {}^t(\omega^1, \omega^2, \omega^3)$ be the tautological form on B. We have arranged things so that $\{\omega^2, \omega^3\} = \pi^*(L_1{}^\perp)$ and $\{\omega^1, \omega^2\} = \pi^*(L_2{}^\perp)$. In particular, the pullback of ω^2 along any section is a contact form.

We now find differential invariants of such G-structures. Since the computation is a little involved, we first give an overview:

(1) We find that $H^{0,2}(\mathfrak{g}) \neq 0$, so we need to normalize the torsion.

(2) We reduce to a subbundle B_1 with structure group $G_1 \subset G$ that preserves our normalization of the torsion.

(3) We calculate that $\mathfrak{g}_1{}^{(1)} \neq 0$, so we need to prolong to $B_1^{(1)}$, a subbundle of the frame bundle over B_1 that has additive structure group $\mathfrak{g}_1{}^{(1)}$.

(4) On $B^{(1)}$, we normalize torsion and reduce to a subbundle with smaller structure group several times, until the structure group $G_2 \subset \mathfrak{g}_1{}^{(1)}$ is such that the connection is unique.

(5) The torsion of the connection on B_2 gives two functionally independent relative differential invariants H_1, H_2. From these we construct tensors on M that are diffeomorphism invariants of the path geometry.

On a first reading of this section, the reader may now wish to skip ahead to page 305, where we interpret the invariants geometrically. In particular, we show that if $H_1 = H_2 = 0$, the path geometry is locally diffeomorphic to the flat case (Example 8.6.12). (Note a slight anomaly of terminology: the flat path geometry does not induce a flat G-structure, since it turns out that B_1 always has torsion.)

Normalizing Torsion. Let $\theta = (\theta^i_j)$ be a connection form on B. We may write

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = -\begin{pmatrix} \theta^1_1 & \theta^2_1 & 0 \\ 0 & \theta^2_2 & 0 \\ 0 & \theta^3_2 & \theta^3_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} T^1_{ij}\omega^i \wedge \omega^j \\ T^2_{ij}\omega^i \wedge \omega^j \\ T^3_{ij}\omega^i \wedge \omega^j \end{pmatrix},$$

where the T^i_{jk}'s are functions on B.

Since every term $T^1_{ij}\omega^i \wedge \omega^j$ involves either ω^1 or ω^2, we can absorb these terms into the connection matrix by adding ω's to θ^1_1 and θ^1_2. The same is true for the third row of torsion, and the parts of the second row which involve ω^2. So, by choosing a new connection $\tilde{\theta}$ we can arrange for all torsion terms except for $T^2_{13}\omega^1 \wedge \omega^3$ to be zero. We conclude that:

Proposition 8.6.17. $\dim H^{0,2}(\mathfrak{g}) = 1$.

The G-structures arising from path geometry will never have $T^2_{13} = 0$, because ω^2 pulls back to be a contact form.

We now use the group G to normalize T^2_{13}. If $g \in G$, then

$$R_g^*\omega = \begin{pmatrix} a^{-1}\omega^1 \\ b^{-1}\omega^2 \\ c^{-1}\omega^3 \end{pmatrix},$$

and it follows that $g \cdot T^2_{13} = \frac{b}{ac} T^2_{13}$.

Let $B_1 \subset B$ be the subbundle where $T^2_{13} \equiv 1$. Its structure group is

$$G_1 = \left\{ \begin{pmatrix} a & e & 0 \\ 0 & ac & 0 \\ 0 & f & c \end{pmatrix} \middle| \, ac \neq 0 \right\}.$$

In our usual abuse of notation, we now let $\theta = (\theta^i_j)$ denote the pullback of the connection form to B_1. Since θ is $\mathfrak{g}_1$-valued when restricted to the fiber of B_1, we have $\theta^2_2 \equiv \theta^1_1 + \theta^3_3 \operatorname{mod}\{\omega^j\}$. New apparent torsion may arise in trying to make θ $\mathfrak{g}_1$-valued. Suppose that

$$\theta^2_2 = \theta^1_1 + \theta^3_3 + a_1\omega^1 + a_2\omega^2 + a_3\omega^3.$$

Then if we let $\phi^1 = \theta^1_1 + a_1\omega^1$ and $\phi^2 = \theta^3_3 + a_3\omega^3$, our new structure equations are

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = -\begin{pmatrix} \phi^1 & \mu^1 & 0 \\ 0 & \phi^1 + \phi^2 & 0 \\ 0 & \mu^2 & \phi^2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ \omega^1 \wedge \omega^3 \\ 0 \end{pmatrix}, \tag{8.10}$$

where $\mu^1 = \theta^1_2$ and $\mu^2 = \theta^3_2$. Thus the apparent torsion is absorbable.

Remark 8.6.18. Our change of notation has an ulterior motive. We want to distinguish the torus $\mathfrak{t} \subset \mathfrak{g}_1$ which is spanned by vectors dual to ϕ_1, ϕ_2. Recall that for a semi-simple Lie algebra, its action on a vector space up to isomorphism is entirely determined by the action of a maximal torus. In particular, in the case of semi-simple $\mathfrak{g}$, to test if a quantity is invariant, it is sufficient to check if it is annihilated by the maximal torus $\mathfrak{t} \subset \mathfrak{g}$ (or equivalently, acted upon trivially by the torus $T \subset G$), something quite easy to check. Recall from Appendix A that such quantities are said to have *weight zero*. In the case of a G-structure with G an arbitrary matrix

Lie group, it is still easy to check if a form is invariant under the action of a maximal torus, which gives a necessary condition for the form to be invariant. We will still say quantities invariant under the action of a maximal torus have *weight zero.*

Example (8.6.16, continued). For the ODE $y'' = F(x, y, y')$, we obtain a section of the G_1-structure by taking

$$\begin{aligned}\underline{\omega}^1 &= dx,\\ \underline{\omega}^2 &= dy - pdx,\\ \underline{\omega}^3 &= dp - F(x, y, p)dx.\end{aligned}$$

In this case, a set of connection forms which satisfy (8.10) is

$$\phi^1 = F_p dx, \qquad \phi^2 = -\phi^1, \qquad \mu^1 = 0, \qquad \mu^2 = -F_y dx. \tag{8.11}$$

Proposition 8.6.19. *The non-uniqueness in the choice of connection satisfying* (8.10) *is*

$$\begin{aligned}\begin{pmatrix}\tilde{\phi}^1\\ \tilde{\mu}^1\end{pmatrix} &= \begin{pmatrix}\phi^1\\ \mu^1\end{pmatrix} + \begin{pmatrix}0 & s_1\\ s_1 & t_1\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},\\ \begin{pmatrix}\tilde{\mu}^2\\ \tilde{\phi}^2\end{pmatrix} &= \begin{pmatrix}\mu^2\\ \phi^2\end{pmatrix} + \begin{pmatrix}t_2 & s_2\\ s_2 & 0\end{pmatrix}\begin{pmatrix}\omega^2\\ \omega^3\end{pmatrix}\end{aligned} \tag{8.12}$$

for some functions s_1, s_2, t_1, t_2. Hence, $\dim \mathfrak{g}_1{}^{(1)} = 4$

Since the connection is not unique, we prolong.

Prolongation. Let $B_1^{(1)}$ be the prolongation of B_1, as described in §8.3. The forms $(\omega^j, \phi^i, \mu^i)$ pulled back to $B_1^{(1)}$ become a basis of the tautological forms on this bundle. We would like fix a connection on $B_1^{(1)}$ and determine the torsion. In order to deduce the specific form of the torsion, we differentiate the semi-basic forms on $B_1^{(1)}$. We already know the derivatives of the ω^j, so we need the derivatives of the ϕ^i and μ^i, which we calculate by computing $0 = d^2\omega^j$:

$$\begin{aligned}0 = d^2\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} &= -\begin{pmatrix}d\phi^1 & d\mu^1 & 0\\ 0 & d\phi^1 + d\phi^2 & 0\\ 0 & d\mu^2 & d\phi^2\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix}\\ &+ \begin{pmatrix}\phi^1 & \mu^1 & 0\\ 0 & \phi^1 + \phi^2 & 0\\ 0 & \mu^2 & \phi^2\end{pmatrix} \wedge \left\{-\begin{pmatrix}\phi^1 & \mu^1 & 0\\ 0 & \phi^1 + \phi^2 & 0\\ 0 & \mu^2 & \phi^2\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} + \begin{pmatrix}0\\ \omega^1 \wedge \omega^3\\ 0\end{pmatrix}\right\}\\ &+ \begin{pmatrix}0\\ -(\phi^1 \wedge \omega^1 + \mu^1 \wedge \omega^2) \wedge \omega^3 + \omega^1 \wedge (\mu^2 \wedge \omega^2 + \phi^2 \wedge \omega^3)\\ 0\end{pmatrix}.\end{aligned}$$

Separating the rows, we have

$$0 = -d^2\omega^1 = (d\phi^1 + \mu^1 \wedge \omega^3) \wedge \omega^1 + (d\mu^1 + \mu^1 \wedge \phi^2) \wedge \omega^2, \tag{8.13}$$

$$0 = -d^2\omega^2 = -(d\phi^1 + d\phi^2) \wedge \omega^2 - \mu^1 \wedge \omega^2 \wedge \omega^3 - \mu^2 \wedge \omega^1 \wedge \omega^2, \tag{8.14}$$

$$0 = -d^2\omega^3 = (d\mu^2 + \mu^2 \wedge \phi^1) \wedge \omega^2 + (d\phi^2 - \mu^1 \wedge \omega^1) \wedge \omega^3. \tag{8.15}$$

The Cartan Lemma A.1.9 applied to (8.13) and (8.15) gives

$$\begin{aligned} \begin{pmatrix} d\phi^1 + \mu^1 \wedge \omega^3 \\ d\mu^1 + \mu^1 \wedge \phi^2 \end{pmatrix} &= \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_2 & \pi_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \\ \begin{pmatrix} d\mu^2 + \mu^2 \wedge \phi^1 \\ d\phi^2 + \mu^2 \wedge \omega^1 \end{pmatrix} &= \begin{pmatrix} \pi_4 & \pi_5 \\ \pi_5 & \pi_6 \end{pmatrix} \wedge \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} \end{aligned} \tag{8.16}$$

for some 1-forms π_i. (The span of the π's, modulo semi-basic forms, will give a connection on $B_1^{(1)}$.) Substituting these equations in (8.14) and reducing mod ω^2 gives

$$\begin{pmatrix} \pi_1 \\ \pi_6 \end{pmatrix} \equiv \begin{pmatrix} -2\mu^2 \\ 2\mu^1 \end{pmatrix} \operatorname{mod} \omega^1, \omega^2, \omega^3.$$

So, π_1 and π_6 are semi-basic for the submersion $B_1^{(1)} \to B_1$. Since, for any c, $\tilde{\pi}_1 = \pi_1 + c\omega^1$ also satisfies (8.16), we may assume $\pi_1 = -2\mu^2 + e\omega^2 + f\omega^3$ for some functions e, f. Moreover, we may assume $e \equiv 0$ by adjusting π_2. Similar considerations for π_6 allow us to rewrite (8.16) as

(8.17)
$$\begin{pmatrix} d\phi^1 \\ d\mu^1 \end{pmatrix} = \begin{pmatrix} 0 & \pi_2 \\ \pi_2 & \pi_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} -\mu^1 \wedge \omega^3 - 2\mu^2 \wedge \omega^1 + E_1\omega^1 \wedge \omega^3 \\ \phi^2 \wedge \mu^1 \end{pmatrix},$$
$$\begin{pmatrix} d\mu^2 \\ d\phi^2 \end{pmatrix} = \begin{pmatrix} \pi_4 & \pi_5 \\ \pi_5 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} \phi^1 \wedge \mu^2 \\ \mu^2 \wedge \omega^1 - 2\mu^1 \wedge \omega^3 + E_2\omega^1 \wedge \omega^3 \end{pmatrix}.$$

The remaining π's are connection forms on $B_1^{(1)}$, and E_1, E_2 are torsion components. However, adding the first and fourth lines and substituting into (8.14) gives $E_1 + E_2 = 0$. So, we will let $E = E_1 = -E_2$.

Since the structure group $\mathfrak{g}_1^{(1)}$ acts additively on the fibers, it acts either additively or trivially on the scalar E. One way of checking how torsion components vary along the fiber is to differentiate the structure equations and mod out by semi-basic forms. In this case, computing $0 = d^2\phi^1$ gives

$$dE \equiv E(\phi^2 - \phi^1) + 2(\pi_2 - \pi_5) \operatorname{mod} \ \omega^1, \omega^2, \omega^3, \tag{8.18}$$

showing that E varies additively when we move along the fibers in the direction dual to $\pi_2 - \pi_5$.

Of course, we can also compute how E varies explicitly. Given a connection on B_1 (i.e., a section of $B_1^{(1)}$), change to a new connection given by

(8.12). Then

$$d\tilde{\phi}^1 = \tilde{\pi}_2 \wedge \omega^2 - \mu^1 \wedge \omega^3 - 2\mu^2 \wedge \omega^1 + (E + 2s_1 - 2s_2)\omega^1 \wedge \omega^3,$$

where

$$\tilde{\pi}_2 = \pi_2 + ds_1 + t_1\omega^3 + 2t_2\omega^1 - s_1(\tilde{\phi}^1 + \tilde{\phi}^2),$$

shows that $2s_1 - 2s_2$ is added to E. So, we may always pick s_1, s_2 so that E becomes zero. Note that this reduces the group to the subgroup where $s_1 = s_2$

Example (8.6.16, continued). One can compute that $E = -F_{pp}$ for the connection forms given by (8.11). We change E to zero by just using the s_1-adjustment, passing to new connection forms

$$\phi^1 = F_p dx + \tfrac{1}{2}F_{pp}(dy - p\,dx), \qquad \mu^1 = \tfrac{1}{2}F_{pp}dx,$$

with ϕ^2 and μ^2 as before.

Exercise 8.6.20: For Example 8.6.14 (Riemannian surfaces), show that

$$\underline{\omega}^1 = \omega^1,\ \underline{\omega}^2 = \omega^2,\ \underline{\omega}^3 = \omega^2_1,\ \phi^1 = 0,\ \phi^2 = 0,\ \mu^1 = -\omega^2_1,\ \mu^2 = K\omega^1$$

(where K is the Gauss curvature) gives a section of $B_1^{(1)}$ for which $E = 0$.

We now restrict to the subbundle on which $E \equiv 0$, where the fiber group now has dimension three. (We don't bother to name it, as we are going to restrict to a further subbundle momentarily.) By (8.18), $\pi_2 - \pi_5 \equiv 0$ modulo $\omega^1, \omega^2, \omega^3$. In fact, we may adjust the π's, while preserving (8.17), so that $\pi_2 - \pi_5$ is a combination of just ω^1 and ω^3. Let

$$\begin{aligned} \pi_2 &= \sigma + (x_1\omega^1 + x_3\omega^3), \\ \pi_5 &= \sigma - (x_1\omega^1 + x_3\omega^3). \end{aligned}$$

The components of $\pi_2 - \pi_5$ are new torsion, which we may normalize to zero. For, calculating $0 = d^2\phi_1 - d^2\phi_2$ gives

$$\left.\begin{aligned} dx_1 &= -\tfrac{3}{2}\pi_4 + x_1(\phi_1 + 2\phi_2) \\ dx_3 &= -\tfrac{3}{2}\pi_3 + x_3(2\phi_1 + \phi_2) \end{aligned}\right\} \operatorname{mod} \omega^1, \omega^2, \omega^3. \tag{8.19}$$

Exercise 8.6.21: Show that, after making a change of connection (8.12) with $s_1 = s_2 = 0$ and $t_1 = \frac{2}{3}x_3$, $t_2 = \frac{2}{3}x_1$, we may assume that $\pi_2 = \pi_5$.

We now restrict our attention to the subbundle $B_2 \subset B_1^{(1)}$ where $x_1 = x_3 = 0$, and we have $\pi_2 = \pi_5 = \sigma$. By (8.19), one sees that now $\pi_3, \pi_4 \equiv 0 \operatorname{mod} \omega^1, \omega^2, \omega^3$.

Exercise 8.6.22: Show that by adjusting σ by a multiple of ω^2 we can arrange for $\pi_3 \wedge \omega^2 = H_2\omega^3 \wedge \omega^2$ for some function H_2.

This normalization forces $\pi_4 \wedge \omega^2 = H_1\omega^1 \wedge \omega^2$ for some function H_1. Note that at this point we have made so many normalizations that we have a uniquely determined coframe of B_2. (A reduction of a G-structure to a unique section, like we have here, is called an *e-structure.*) Before discussing this coframe, we return to our example:

Example (8.6.16, continued). Substituting our connection forms into (8.17) shows that we may take

$$\pi_2 = (F_{py} + \tfrac{1}{2}D_xF_{pp})\omega^1 + \tfrac{1}{2}F_{ppp}\omega^3, \qquad \pi_5 = F_{py}\omega^1,$$

where $D_x(g(x,y,p)) = g_x + p\,g_y + g\,g_p$. These give us the values for x_1 and x_3. We make the change of connection described in Exercise 8.6.21, resulting in

$$\mu_1 = \tfrac{1}{2}F_{pp}\omega^1 + \tfrac{1}{6}F_{ppp}\omega^2, \qquad \mu_2 = -F_y\omega^1 + \left(\tfrac{1}{6}D_xF_{pp} - \tfrac{2}{3}F_{py}\right)\omega^3,$$

with ϕ^1 and ϕ^2 the same as before. Now we recalculate, obtaining

$$\sigma = \left(\tfrac{1}{3}F_{py} + \tfrac{1}{6}D_xF_{pp}\right)\omega^1 + a\omega^2 + \tfrac{1}{3}F_{ppp}\omega^3$$

for arbitrary a. We choose a so that $\pi_3 \equiv 0$ modulo ω^2, ω^3, and obtain

$$\begin{aligned} H_1 &= F_{yy} + \tfrac{1}{6}\left(D_x^2F_{pp} - 4D_xF_{py} + F_p(4F_{py} - D_xF_{pp}) - 3F_yF_{pp}\right) \\ H_2 &= \tfrac{1}{6}F_{pppp}. \end{aligned} \tag{8.20}$$

Our uniquely determined coframe on B_2 is ω^1, ω^2, ω^3, ϕ^1, ϕ^2, μ^1, μ^2, σ. It satisfies the structure equations (8.10) and

$$\begin{aligned} d\phi^1 &= -\mu^1 \wedge \omega^3 - 2\mu^2 \wedge \omega^1 + \sigma \wedge \omega^2, \\ d\phi^2 &= 2\mu^2 \wedge \omega^3 + \mu^2 \wedge \omega^1 + \sigma \wedge \omega^2, \\ d\mu^1 &= \phi^2 \wedge \mu^1 + \sigma \wedge \omega^1 + H_2\omega^3 \wedge \omega^2, \\ d\mu^2 &= \phi^1 \wedge \mu^2 + \sigma \wedge \omega^3 + H_1\omega^1 \wedge \omega^2. \end{aligned} \tag{8.21}$$

By differentiating the above equations, we deduce that

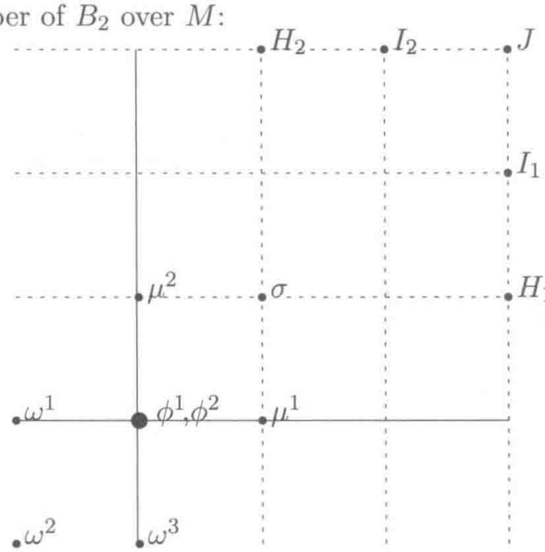
$$d\sigma = (\phi^1 + \phi^2) \wedge \sigma - \mu^1 \wedge \mu^2 + (I_1\omega^1 + I_2\omega^3) \wedge \omega^2 \tag{8.22}$$

for some functions I_1, I_2. Differentiating (8.21) and (8.22) also gives

$$\begin{aligned} dH_1 &\equiv -I_1\omega^3 + H_1(\phi^2 + 3\phi^1) \mod \omega^1, \omega^2, \\ dH_2 &\equiv -I_2\omega^1 + H_2(\phi^1 + 3\phi^2) \mod \omega^2, \omega^3, \\ dI_1 &\equiv I_1(3\phi^1 + 2\phi^2) - H_1\mu^1 + J\omega^3 \mod \omega^1, \omega^2, \\ dI_2 &\equiv I_2(2\phi^1 + 3\phi^2) + H_1\mu^2 + J\omega^1 \mod \omega^2, \omega^3 \end{aligned} \tag{8.23}$$

for some function J.

Interpreting the Invariants. The equations (8.23) help us complete a *weight diagram* (see [**52**]), which shows how our various forms and functions scale along the fiber of B_2 over M:



In this diagram, weights are assigned according to how objects scale with respect to directions in the maximal torus dual to ϕ^1 and ϕ^2 respectively; the large dot represents the origin.

The reader familiar with such diagrams will recognize that if $H_1, H_2 = 0$ we have the weight diagram for the adjoint representation of $\mathfrak{sl}_3$, and will have already guessed the following proposition:

Proposition 8.6.23. *The relative invariants H_1 and H_2 are identically zero if and only if the path geometry is locally diffeomorphic to the flat model of straight lines in the plane.*

Proof. For the first part, notice that for the ODE $y'' = 0$, we obtain $H_1 = H_2 = 0$ from the formulas (8.20). For the second part, let

$$\psi = \begin{pmatrix} -\frac{1}{3}(2\phi^1 + \phi^2) & -\mu^2 & \sigma \\ \omega^1 & \frac{1}{3}(\phi^1 - \phi^2) & \mu^1 \\ \omega^2 & \omega^3 & \frac{1}{3}(\phi^1 + 2\phi^2) \end{pmatrix}.$$

Note that ψ takes values in the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. In general,

$$d\psi + \psi \wedge \psi = \begin{pmatrix} 0 & -H_1\omega^1 \wedge \omega^2 & (I_1\omega^1 + I_2\omega^3) \wedge \omega^2 \\ 0 & 0 & H_2\omega^3 \wedge \omega^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

If H_1 and H_2 are identically zero, then I_1 and I_2 are also zero by (8.23). Since $d\psi + \psi \wedge \psi = 0$, by Theorem 1.6.10 there exists, in a neighborhood U of any point on B_2, a local diffeomorphism $g : U \to SL(3, \mathbb{R})$ such that $g^{-1}dg = \psi$. Moreover, the fibers of submersion to M, which are annihilated

by the ω's, are carried to right cosets of the subgroup P of upper triangular matrices in $SL(3,\mathbb{R})$. This means that M is locally diffeomorphic to the flag variety $\mathbb{F}_{12}$ from Example 8.6.12. $\square$

Since neither H_1 nor H_2 has weight zero, they do not descend to be invariant functions on M. We can however pair them with quantities of opposite weights to obtain forms of weight zero. Consider the quartic forms $h_1 := H_1(\omega^1)^3\omega^3$ and $h_2 := H_2\omega^1(\omega^3)^3$, which are of weight zero. These still don't descend to sections of S^4T^*M, because of the off-diagonal part of the action of G_1 (i.e., adding ω^2 to ω^1 or ω^3). However, they are well-defined sections of S^4W^*, where $W = \omega^{2\perp} \subset TM$. Recall that $M = \mathbb{P}(TX)$ and that ω^2 corresponds to the contact form, so this bundle has a natural geometric interpretation.

Consider Example 8.6.14 of paths arising from a Riemannian metric. Using Proposition 8.6.23, we can prove

Theorem 8.6.24 (Bonnet). *The only metrics inducing the flat path geometry on a surface X are metrics of constant Gauss curvature.*

Proof. It is easy to check that the coframe in Exercise 8.6.20 gives a section of B_2. Differentiating the structure equations of $\mathcal{F}_{on}(X)$ gives $dK \equiv 0$ modulo ω^1, ω^2. Let

$$dK = K_1\omega^1 + K_2\omega^2. \tag{8.24}$$

Using $\sigma = -K\omega^2$, we compute that $H_1 = K_2$ and $H_2 = 0$. Furthermore, differentiating (8.24) shows that $dK_2 \equiv -K_1\omega_1^2$ modulo ω^1, ω^2. Hence, H_1 is identically zero if and only if K is constant. $\square$

Remark 8.6.25. In fact, the same result is true for path geometries of arbitrary dimensions: the only Riemannian metrics inducing the flat path geometries are those of constant sectional curvature.

Proposition 8.6.26. *A path geometry is locally equivalent to a projective structure if and only if $H_2 \equiv 0$.*

Proof. Since every path geometry is locally equivalent to one arising from a second-order ODE, we can use the formulas (8.20) in that case to give interpretations to each of H_1 and H_2. For example, $H_2 = 0$ implies that our system of paths is locally equivalent to the integral curves of an ODE of the form

$$y'' = A(y')^3 + B(y')^2 + C(y') + D,$$

where A, B, C, D are functions of x and y. Now apply 8.4.8 $\square$

Remark 8.6.27. If one really wanted to find the explicit integral curves of an ODE giving rise to a flat set of paths, it would be rather difficult to solve,

because although one knows abstractly that one has the flat case, actually finding the transformation of integral curves can be quite difficult.

If $H_1 \equiv 0$ and $H_2 \neq 0$, then up to a few exceptions the equation $y'' = F(x, y, p)$ is solvable by quadrature. (In other words, the differential equations have a solvable symmetry group.) The idea is that differentiating H_2 and I_2 yields scalar invariants that are constant along the paths, yielding first integrals for the ODE. We explain this in more detail, following [**23**]:

Assume that $H_1 \equiv 0$. Then $I_1 \equiv 0$ and $J \equiv 0$ from (8.23), and we have

$$d\begin{pmatrix} I_2 \\ H_2 \end{pmatrix} \equiv \begin{pmatrix} 2\phi^1 + 3\phi^2 & \mu^2 \\ -\omega^1 & \phi^1 + 3\phi^2 \end{pmatrix} \begin{pmatrix} I_2 \\ H_2 \end{pmatrix} \mod \omega^2, \omega^3. \tag{8.25}$$

This implies that $\mathcal{P} = (I_2\omega^2 + H_2\omega^3) \otimes (\omega^2 \wedge \omega^3)$ is a well-defined tensor on the path space $Z = M/L_1$. (Note that the weight diagram indicates that $\mathcal{P}$ is invariant under the maximal torus.) To see this, suppose we set $P_2 = I_2$, $P_3 = H_2$. Then we may write (8.25) as

$$dP_a = (\psi_{ab} + (\operatorname{tr}\Psi)\delta_{ab})\, P_b + P_{ab}\omega^b, \tag{8.26}$$

for some functions P_{ab}, where we use index ranges $2 \leq a, b \leq 3$ and we have set

$$\Psi = (\psi_{ab}) = \begin{pmatrix} \phi^1 + \phi^2 & \mu^2 \\ -\omega^1 & \phi^2 \end{pmatrix}.$$

Differentiating (8.26) allows us to extract similar tensorial invariants from the derivatives P_{ab}. Namely, let

$$Q_2 = P_2(P_{23} - 4P_{32}) + 3P_3P_{22}, \qquad Q_3 = P_3(4P_{23} - P_{32}) - 3P_2P_{33}.$$

Then

$$dQ_a = (\psi_{ab} + 3(\operatorname{tr}\Psi)\delta_{ab})\, Q_b + Q_{ab}\omega^b$$

shows that $\mathcal{Q} = (Q_2\omega^2 + Q_3\omega^3) \otimes (\omega^2 \wedge \omega^3)^3$ gives a well-defined section of $T^*Z \otimes L^3$, where $L = \Lambda^2(T^*Z)$. The wedge product $\mathcal{P} \wedge \mathcal{Q}$ is therefore an invariant taking value in the line bundle L^5.

Continuing to differentiate leads to more line-bundle invariants. For example, if we set

$$R_2 = Q_2(Q_{23} - 10Q_{32}) + 9Q_3Q_{22}, \qquad R_3 = Q_3(10Q_{23} - Q_{32}) - 9Q_2Q_{33},$$

then $\mathcal{R} = (R_2\omega^2 + R_3\omega^3) \otimes (\omega^2 \wedge \omega^3)^7$ is well-defined on Z. If $\mathcal{P} \wedge \mathcal{Q}$ is nonzero, then the quotient

$$\frac{(\mathcal{P} \wedge \mathcal{R})^5}{(\mathcal{P} \wedge \mathcal{Q})^9}$$

gives a well-defined function on path space, and therefore is a first integral for the ODE.

Exercise 8.6.28: Verify that $H_1 \equiv 0$ for the path geometry associated to the ODE

$$y'' = \frac{(y')^2(1+(y')^2)}{y(1+\sqrt{1+(y')^2})}. \tag{8.27}$$

Then calculate the invariants $\mathcal{P}$ and $\mathcal{Q}$ for this geometry.

Remark 8.6.29. If one is working in the holomorphic category, one can obtain much stronger global results. Recall that infinitesimal deformations of submanifolds are roughly given by sections of their normal bundles. Let Z be a complex compact surface and let $C \subset Z$ be a holomorphic curve isomorphic to $\mathbb{CP}^1$. We will say C has *self-intersection one* if any small deformation of C will intersect C in exactly one point. Thus its deformations are to first order the same as those of a line in the projective plane.

Theorem 8.6.30 (Hitchin [**78**]). *Let Z be a complex surface equipped with a path geometry whose paths are rational curves with self-intersection one. Then $H_2 \equiv 0$, i.e., the path geometry comes from a projective structure. Moreover, any path geometry in a compact complex surface Z with $H_2 \equiv 0$ has rational curves with self-intersection one as its paths.*

8.7. Frobenius and generalized conformal structures

This section is an overview of some recent developments. It is less detailed, and the reader is referred to the articles mentioned below for more information.

Motivated by considerations in integral geometry, Goncharov, in [**62**], defined a *Frobenius structure* on a manifold M as follows. Let W be a vector space with $\dim W = \dim M = n$, fix $k < n$ and let $F \subset G(k, W)$ be a submanifold. An F-Frobenius structure on M consists of, for each $x \in M$, a submanifold $F_x \subset G(k, T_xM)$ isomorphic to F and such that the F_x vary smoothly as one varies x.

For example, if F is a point, an F-Frobenius structure corresponds to a distribution of k-planes on M. If M is equipped with a Lorenztian metric, the null cone induces a Frobenius structure in $G(1, TM)$ by taking F_x to be the set of null lines in T_xM.

The *flat Frobenius structure* on $W \simeq \mathbb{R}^n$ is the Frobenius structure obtained by translating F around using the identification $T_xW \simeq W$. A Frobenius structure on M is *locally flat* on an open subset $U \subset M$ if there exists a diffeomorphism $\phi : U \to W$ preserving the Frobenius structures, i.e., for all $x \in U$, the map $\tilde{\phi}_{*x} : G(k, T_xM) \to G(k, W)$ induced by $\phi_{*x} : T_xM \to T_xW \simeq W$ induces an isomorphism $F_x \to F$.

Let $G \subset GL(W)$ be a semi-simple matrix Lie group and let M be equipped with a G-structure. Let $F = G/H \subset W$ be the orbit of a highest weight vector (see §A.6). Then M is equipped with an F-Frobenius structure with $k = 1$ via $\mathbb{P}F \subset G(1, W)$. Goncharov calls this situation–a Frobenius structure induced by a homogeneous subvariety of the Grassmannian–an *$\mathcal{F}$-structure*. Note that here the notions of flatness for Frobenius and G-structures coincide. Note also that if $\mathbb{P}F$ is $\mathbb{P}^{k-1}$-uniruled, then one also gets an induced $\mathcal{F}$-structure via $\mathbb{F}_{k-1}(\mathbb{P}F) \subset G(k, W)$.

$\mathcal{F}$-structures of particular interest arise when there is a homogeneous model space equipped with some $F = G/H$-Frobenius structure. That is, there exist a Lie group S and a homogeneous space $M = S/G$ such that the isotropy action of G on $W = T_{[e]}M$ induces a G-structure on M and an $F = G/H$-Frobenius structure, where G/H is the orbit of a highest weight vector.

For example, let $S = SU(p+q)$, $G = S(U(p) \times U(q))$, so $S/G = G(p, \mathbb{C}^{p+q})$ is the usual Grassmannian. Then $F = \mathrm{Seg}(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1})$ is the Segre variety (see §3.3).

If one is interested in cases where $M = S/G$ is a compact complex homogeneous space and where F_x is linearly nondegenerate, then we are reduced to M being (a quotient of) a compact Hermitian symmetric space (CHSS for short); see [**110**]. Following tradition, such cases are called *S-structures*, and Goncharov calls the corresponding $\mathcal{F}$-structures *generalized conformal structures associated to CHSS*. The first example of such are the generalized conformal structures modeled on the Grassmannians, which were first introduced by Akivis [**3**] in the study of higher-dimensional webs. The G-structure on a CHSS $M = S/G$ is actually flat (see [**62**], Proposition 4.1).

The CHSS are products of the following irreducible ones: the Grassmannian $G(k, n)$, the spinor variety $\mathbb{S}_n$ and the Lagrangian Grassmannian $G_{Lag}(n, 2n)$ (see §3.3), the quadric hypersurfaces Q^n, the complexified Cayley plane $\mathbb{OP}^2$ (see Exercise 3.6.9.3), and E_7/P_7 (see [**112**] for a geometric description of this last variety). Except for the Grassmannians, the semisimple part of the isotropy group acting on $T_{[e]}M$ is simple, and in the case of the Grassmannians it is the product of two simple groups. Thus in this situation one can apply the Bott-Borel-Weil theorem (see [**9**]). Goncharov [**62**] calculated:

1. For $M = G(k, k+l)$, a Grassmannian, and $\mathfrak{g} = \mathfrak{a}_{k-1} + \mathfrak{a}_{l-1}$:

1a. If $k, l > 2$, then all $H^{i,2}(\mathfrak{g})$ are zero except for $H^{0,2}(\mathfrak{g})$, which is the direct sum of two irreducible factors.

1b. If $k = 2 < l$, then $H^{0,2}(\mathfrak{g})$ and $H^{1,2}(\mathfrak{g})$ are both nonzero and irreducible, and all other $H^{i,2}(\mathfrak{g})$ are zero.

1c. If $k = l = 2$, so $M = Q^4 = G(2,4)$, then all $H^{i,2}(\mathfrak{g})$ are zero except for $H^{1,2}(\mathfrak{g})$, which is the direct sum of two irreducible factors.

2. For $M = Q^n$ and $\mathfrak{g} = \mathfrak{so}(n)$:

2a. If $n > 4$, then $H^{1,2}$ is nonzero and irreducible, and all $H^{i,2}(\mathfrak{g})$ are zero.

2b. If $n = 3$, then $H^{2,2}$ is nonzero and irreducible, and all $H^{i,2}(\mathfrak{g})$ are zero.

3. In all other cases, the only nonzero group is $H^{0,2}$, which is irreducible.

Thus, except for cases 1b,1c and 2, any S-structure that is 1-flat is flat!

The following result of Ochiai generalizes Liouville's theorem about holomorphic mappings $\mathbb{P}^1 \to \mathbb{P}^1$:

Theorem 8.7.1 (Ochiai [**125**]). *Let M be a simply-connected compact complex manifold with a flat S-structure. Then M is biholomorphic to the corresponding compact Hermitian symmetric space.*

Thus, if one has a compact complex manifold M equipped with an S-structure, if one can prove the S structure is flat, then the universal cover of M is completely identified.

Kobayashi and Ochiai [**95**], by using Chern classes, proved that if a complex manifold M admits a Kähler-Einstein metric, then any S-structure on it is flat.

Hwang and Mok [**81**] showed that if M is Fano, then any S-structure on M is flat. In fact they proved a more general result: call a complex manifold *uniruled* if through each $x \in M$ there exists a rational curve (i.e., an immersed $\mathbb{P}^1$) in M containing x. (This generalizes the notion of linearly uniruled projective varieties introduced in Chapter 3.) Fano manifolds are uniruled. Hwang and Mok show that an S-structure on a uniruled projective manifold is necessarily flat. To prove this, they first show that there must be rational curves tangent to the induced F-structure. They then show that all the cohomology groups vanish at once by pulling a certain vector bundle back to $\mathbb{P}^1$ and studying its sections. Note that the Hwang-Mok theorem follows from global properties of one-dimensional integral manifolds of the F-structure.

In contrast, Goncharov [**62**] proves a flatness theorem using local properties. He shows that if there exist local integral manifolds of the F-structure associated to an S-structure of maximal dimension, then the F-structure is necessarily flat. (In the case of the Grassmannians, one needs two types of integral manifolds.)

Appendix A

Linear Algebra and Representation Theory

In this appendix we briefly discuss tensors (§A.1), matrix Lie groups (§A.2) and Lie algebras (§A.4), complex vector spaces and complex structures (§A.3), the octonions and the exceptional group G_2 (§A.5), Clifford algebras and spin groups (§A.7), and outline some rudiments of representation theory (§A.6).

Unless otherwise noted, V, W are real vector spaces of dimensions n and m, $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are their bases, and we use the index ranges $1 \le i, j \le n$ and $1 \le s, t \le m$.

A.1. Dual spaces and tensor products

A map $f : V \to W$ is *linear* if $f(v + v') = f(v) + f(v')$ and $f(kv) = kf(v)$ for all $v, v' \in V$ and $k \in \mathbb{R}$.

Definition A.1.1. The *dual space* of V, denoted by V^*, is the space of all linear maps $f : V \to \mathbb{R}$. It is a vector space under the operations of scalar multiplication and addition of maps.

Exercises A.1.2:

1. Let $\alpha^i \in V^*$ be defined by $\alpha^i(v_j) = \delta^i_j$. Show that $\alpha^1, \dots, \alpha^n$ is a basis for V^*, called the *dual basis* to $v_1, \dots, v_n$. In particular, $\dim V^* = n$.
2. Define, in a coordinate-free way, an isomorphism $V \to (V^*)^*$. (Note that these spaces would not necessarily be isomorphic if V were replaced by an infinite-dimensional vector space.)

Geometric Aside. Given V, one can form the associated projective space $\mathbb{P}V$, which is the space of all lines thorough the origin in V. A vector $\alpha \in V^*$ determines a codimension-one linear subspace $\ker \alpha \subset V$, so that $\mathbb{P}V^*$ may be interpreted as the space of hyperplanes through the origin in V.

Definition A.1.3. Let $\mathrm{Hom}(V, W)$ denote the space of linear maps from V to W. Like V^*, this is a vector space under addition of maps. The space of linear maps from V to V, called *endomorphisms*, is denoted by $\mathrm{End}(V)$.

We may think of $\mathrm{Hom}(V, W)$ as the space of W-valued linear functions on V, and when doing so we denote it by $V^* \otimes W$, the *tensor product* of V^* and W. Similarly, we define $V \otimes W$ as the space of W-valued linear functions on V^*.

Given $\alpha \in V^*$ and $w \in W$, we may define an element of $\mathrm{Hom}(V, W)$, denoted by $\alpha \otimes w$, to be the map

$$v \mapsto \alpha(v) w.$$

We call such an element *decomposable.*

Exercises A.1.4:

1. Show that the decomposable elements in $V \otimes W$ span $V \otimes W$. More precisely, show that the nm vectors $\{v_i \otimes w_s\}$ span $V \otimes W$.
2. After having fixed bases, define an explicit isomorphism between $V^* \otimes W$ and the space of $n \times m$ matrices.
3. Show that the decomposable elements $V^* \otimes W$ are exactly those represented by *rank one* matrices. More generally, show that the *rank* of an element of $V \otimes W$ is well-defined and agrees with the rank of the associated matrix (with respect to any choices of bases).
4. Given $f \in \mathrm{Hom}(V, W)$ we define $f^t \in \mathrm{Hom}(W^*, V^*)$, called the *transpose* or *adjoint* of f, by $f^t(\beta)(v) = \beta(f(v))$. (If we choose dual bases, the matrix representative of f^t is the transpose of that of f.) Show that transpose defines a vector space isomorphism $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(W^*, V^*)$.

Definition A.1.5. Let $V_1, \ldots, V_k$ be vector spaces. A function

$$f : V_1 \times \ldots \times V_k \to W \tag{A.1}$$

is *multilinear* if it is linear with respect to addition and scalar multiplication in each factor V_ℓ. We denote this space of multilinear functions by $V_1^* \otimes V_2^* \otimes \ldots \otimes V_k^* \otimes W$. In particular, $V^{*\otimes 2} = V^* \otimes V^*$ is the space of real-valued bilinear forms on V, and $V^{*\otimes 3} = V^* \otimes V^* \otimes V^*$ is the space of trilinear forms, etc.

If $\beta_1 \in V_1^*, \ldots, \beta_k \in V_k^*$, we define $\beta_1 \otimes \ldots \otimes \beta_k \in V_1^* \otimes V_2^* \otimes \ldots \otimes V_k^*$ by

$$\beta_1 \otimes \ldots \otimes \beta_k(u_1, \ldots, u_k) = \beta_1(u_1) \ldots \beta_k(u_k)$$

and call an element of $V_1^*\otimes V_2^*\otimes\ldots\otimes V_k^*$ *decomposable* if it may be written in this way.

Exercises A.1.6:

1. Show that $(\alpha, v)\mapsto\alpha(v)$ is multilinear from $V^*\times V$ to $\mathbb{R}$.
2. Show that the space of multilinear functions (A.1) is a vector space, and determine its dimension.
3. Show that $V_1^*\otimes V_2^*\otimes\ldots\otimes V_k^*$ is spanned by its decomposable vectors. ◎
4. Given $\alpha\in V^*$, $\beta\otimes W^*$, let $\alpha\otimes\beta(v\otimes w)=\alpha(v)\beta(w)$. Show that this defines an isomorphism $V^*\otimes W^*\cong(V\otimes W)^*$. Thus, $V\otimes W$ may be thought of as the set of linear maps from V^* to W, the set of linear maps from W^* to V, the set of bilinear maps from $V^*\times W^*$ to $\mathbb{R}$, or as the dual space of $V^*\otimes W^*$.
5. Let $V^{\otimes k}$ denote the k-fold tensor product of V with itself. Show that this is the dual space of $V^{*\otimes k}$.

Remark A.1.7. One may define the *rank* of an element $X\in V_1^*\otimes V_2^*\otimes\ldots\otimes V_k^*$ to be the minimal number r such that $X=\sum_{u=1}^r z_u$ with each z_u decomposable. It turns out that the rank is quite subtle for $k>2$. In fact the maximal rank of an element of even a triple tensor product is not known even for low-dimensional vector spaces. Such open questions go under the name *Waring problems*. See Chapter 3 for a geometric generalization.

An open question of importance to computer science is the following: Let A, B, C be vector spaces, and consider the matrix multiplication operator m that composes a linear map from A to B with a linear map from B to C to obtain a linear map from A to C. Let $V_1=A^*\otimes B$, $V_2=B^*\otimes C$, $V_3=A^*\otimes C$. We have $m\in V_1^*\otimes V_2^*\otimes V_3$. Then the open question is, determine the rank of m. For an overview of this problem, see [**144**].

Symmetric and skew-symmetric tensors. The tensor product $\otimes$ is not symmetric; even in $V\otimes V$, $v_1\otimes v_2\neq v_2\otimes v_1$.

Consider $V^{\otimes 2}=V\otimes V$ with basis $\{v_i\otimes v_j|1\leq i,j\leq n\}$. The subspaces defined by

$$\begin{aligned}
S^2V &:= \operatorname{span}\{v_i\otimes v_j+v_j\otimes v_i,\ 1\leq i,j\leq n\}=\operatorname{span}\{v\otimes v\,|v\in V\}\\
&=\{X\in V\otimes V\mid X(\alpha,\beta)=X(\beta,\alpha)\ \forall\alpha,\beta\in V^*\}\\
\Lambda^2V &:= \operatorname{span}\{v_i\otimes v_j-v_j\otimes v_i,\ 1\leq i,j\leq n\}\\
&=\operatorname{span}\{v\otimes w-w\otimes v\,|v,w\in V\},\\
&=\{X\in V\otimes V\mid X(\alpha,\beta)=-X(\beta,\alpha)\ \forall\alpha,\beta\in V^*\}
\end{aligned}$$

are respectively the spaces of *symmetric* and *skew-symmetric* 2-tensors of V. For arbitrary $v_1, v_2\in V$, we define $v_1\circ v_2=v_1v_2:=(v_2\otimes v_2+v_2\otimes v_1)\in S^2V$

and

$$v_1 \wedge v_2 := v_1 \otimes v_2 - v_2 \otimes v_1 \in \Lambda^2 V. \tag{A.2}$$

(Sometimes a factor of $\frac{1}{2}$ is inserted into the definitions of $v_1 \circ v_2$ and $v_1 \wedge v_2$.) More generally,

$$v_1 \wedge \cdots \wedge v_k = \mathcal{A}(v_1 \otimes \cdots \otimes v_k) := \sum_{\sigma} (\operatorname{sgn} \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \tag{A.3}$$

where σ runs over all permutations and $\operatorname{sgn} \sigma = \pm 1$ is the sign of σ. The product $v_1 \wedge \cdots \wedge v_k$ is called the *wedge product* of vectors $v_1, \ldots, v_k$.

Exercises A.1.8:
1. Show that the two definitions for $S^2 V$ are equivalent, as well as the two definitions for $\Lambda^2 V$. Show that

$$V \otimes V = S^2 V \oplus \Lambda^2 V. \tag{A.4}$$

Note that the coordinate-free definition implies that this decomposition is preserved under linear changes of coordinates.
2. Give similar definitions for the totally symmetric k-tensors $S^k V \subset V^{\otimes k}$, and the alternating k-tensors $\Lambda^k V \subset V^{\otimes k}$. ⊚

We often think of $S^k V^*$ as the space of homogeneous polynomials of degree k on V.
3. Given $\alpha \in \Lambda^j V, \beta \in \Lambda^l V$, we define $\alpha \wedge \beta \in \Lambda^{j+l} V$ by restricting (A.3) to $\Lambda^j V \otimes \Lambda^l V \subset \Lambda^k V$ when $k = j + l$. Show that $\beta \wedge \alpha = (-1)^{jl} \alpha \wedge \beta$.
4. Show that $\dim S^k V = \binom{n+k-1}{k}$ and $\dim \Lambda^k V = \binom{n}{k}$. In particular, $\Lambda^n V \simeq \mathbb{R}$, $\Lambda^l V = 0$ for $l > n$, and $S^3 V \oplus \Lambda^3 V \neq V^{\otimes 3}$. ⊚
5. In this exercise, we refine the decomposition $V^{\otimes 3} = V \otimes (V \otimes V) = V \otimes S^2 V \oplus V \otimes \Lambda^2 V$ induced by (A.4). Let

$$\rho(v_1 \otimes v_2 \otimes v_3) = \tfrac{1}{3}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_3 \otimes v_1),$$

and extend this to a linear map $\rho : V^{\otimes 3} \to V^{\otimes 3}$.
(a) Show that ρ restricts to be a projection $V \otimes S^2 V \to S^3 V$ and let $K_S \subset V \otimes S^2 V$ be its kernel.
(b) Similarly, show that ρ restricts to be a projection $V \otimes \Lambda^2 V \to \Lambda^3 V$ and let K_Λ be its kernel.
(c) Conclude that $V^{\otimes 3} = S^3 V \oplus K_S \oplus \Lambda^3 V \oplus K_\Lambda$.
(d) Show that $\dim K_S = \dim K_\Lambda$.

Lemma A.1.9 (Cartan Lemma). *Let $v_1, \ldots, v_k$ be linearly independent elements of V and let $w_1, \ldots, w_k$ be elements of V such that $w_1 \wedge v_1 + \ldots + w_k \wedge v_k = 0$. Then there exist scalars $h_{ij} = h_{ji}, 1 \leq i, j \leq k$, such that $w_i = \sum_j h_{ij} v_j$.*

Exercise A.1.10: Prove the lemma.

Exercise A.1.11: Let $v_1, \dots, v_k$ be as above and let $R_j \in \Lambda^2 V$ be such that $R_j \wedge v^j = 0$. (Here, we use the convention that summation is implied over repeated indices.) Show that $R_j = R_{jkl} v^k \wedge v^l$ for scalars R_{jkl} such that $R_{jlk} = -R_{jkl}$ and $R_{jkl} + R_{klj} + R_{ljk} = 0$.

Induced linear maps. Given $\alpha \in \mathrm{End}(V)$, define maps $\alpha^{\otimes k} : V^{\otimes k} \to V^{\otimes k}$ induced by α as follows. On decomposable elements, let

$$v_1 \otimes v_2 \otimes \dots \otimes v_k \mapsto \alpha(v_1) \otimes \alpha(v_2) \otimes \dots \otimes \alpha(v_k),$$

and extend by linearity. Note that $\alpha^{\otimes k}$ preserves the subspaces $S^k V$ and $\Lambda^k V$.

In particular, the induced map $\alpha^{\otimes n} : \Lambda^n V \to \Lambda^n V$ is multiplication by some scalar. Call this number $\det(\alpha)$, the *determinant* of α. Geometrically, if $P \subset V$ is a parallelepiped of dimension n with one vertex the origin, then $\det(\alpha) = \mathrm{vol}(\alpha(P))/\mathrm{vol}(P)$, where vol is any volume form compatible with the linear structure.

Exercise A.1.12: Show that $\det(\alpha)$ equals the determinant of the square matrix which represents α with respect to a given basis.

Interior products. For $x \in V$, define the *interior product* $x \lrcorner : \Lambda^{p+1} V^* \to \Lambda^p V^*$ by

$$(x \lrcorner \phi)(v_1, \dots, v_p) := \phi(x, v_1, \dots, v_p), \qquad \phi \in \Lambda^{p+1} V^*.$$

More generally, for $z \in \Lambda^p V$ define $z \lrcorner : \Lambda^q V^* \to \Lambda^{q-p} V^*$.

Exercise A.1.13: Show that $x \lrcorner$ is the adjoint of the linear map $x \wedge : \Lambda^p V \to \Lambda^{p+1} V$ given by wedging with x.

Induced inner products and the $*$-operator. An inner product $\langle , \rangle$ on V induces an inner product on V^*, as follows: take any orthonormal basis of V and declare the dual basis to be orthonormal. Alternatively, fix an orthonormal basis $e_1, \dots, e_n$ for V and define $\langle \alpha, \beta \rangle = \sum_i \alpha(e_i)\beta(e_i)$ for $\alpha, \beta \in V^*$.

Exercise A.1.14: Verify that these two definitions of induced inner products are both well-defined and agree.

One may induce inner products on all tensor spaces constructed from V and V^*. For example, the inner product on $V \otimes V$ is given by $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$. However, since our definition (A.3) for the wedge product produces $k!$ terms, we normalize the inner product on $\Lambda^k V$ by dividing the inner product inherited from $V^{\otimes k}$ by $k!$. With this normalization, the inner product satisfies *Hadamard's inequality*

$$|a \wedge b| \leq |a||b|. \tag{A.5}$$

Let $\Omega \in \Lambda^n V$ be a volume form with unit length with respect to this renormalized inner product. For $\alpha \in \Lambda^k V$, we define $*\alpha \in \Lambda^{n-k}V$ by requiring that

$$\beta \wedge *\alpha = \langle \alpha, \beta \rangle \Omega \qquad \forall \beta \in \Lambda^k V.$$

Exercise A.1.15: If $e_1, \ldots, e_n$ is an orthonormal basis, show that $\Omega = e_1 \wedge \ldots \wedge e_n$ has unit length, and calculate $*e_j$. When $n = 2$, are there vectors v such that $*v = v$? What about over $\mathbb{C}$?

Grassmannians and flag varieties. An element of $\Lambda^k V$ is called *decomposable* if it may be written as $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ for some $v_1, \ldots, v_k \in V$. By identifying $[v_1 \wedge \ldots \wedge v_k] \in \mathbb{P}(\Lambda^k V)$ with the k-plane spanned by $v_1, \ldots, v_k$, the projectivization of the set of decomposable elements becomes the space of all k-planes through the origin in V, called the *Grassmannian* of k-planes in V, and is denoted $G(k, V) \subset \mathbb{P}(\Lambda^k V)$.

Exercises A.1.16:

1. Show that $G(k, V)$ is well-defined, i.e., if $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are two bases for a k-plane $E \subset V$, then $[v_1 \wedge \ldots \wedge v_k] = [w_1 \wedge \ldots \wedge w_k] \in \mathbb{P}(\Lambda^k V)$.
2. For $\phi \in \Lambda^2 V$, show that $[\phi] \in G(2, V)$ iff $\phi \wedge \phi = 0$.
3. Given the basis $v_1, \ldots, v_n$ for V, define coordinates x^{ij} on $\Lambda^2 V$, where $x^{ji} = -x^{ij}$, such that elements are given by $z = \sum_{i<j} x^{ij} v_i \wedge v_j$. Using exercise 2, calculate the equations defining $G(2, V)$ when $n = 4$, $n = 5$ and in the general case. These equations go back to Plücker.
4. Define

$$F_{12} = \{[v \otimes E] \in \mathbb{P}(V \otimes \Lambda^2 V) \mid [E] \in G(2, V) \text{ and } v \wedge E = 0\}.$$

Show that $F_{12} \subset \mathbb{P}(K_\Lambda)$ and that it admits the geometric interpretation of the space of *flags*, $l \subset E \subset V$, where l is a line through the origin and E is a two-plane. More generally, one can define the space $F_{a_1, \ldots, a_r}$ of flags $E_1 \subset \ldots \subset E_r \subset V$ such that $\dim E_j = a_j$. See §3.3 for more details.

A.2. Matrix Lie groups

Let $GL(V) \subset \mathrm{End}(V)$ denote the group of all invertible linear maps. Let G be a *Lie group*, i.e., a C^∞-manifold that has a group structure compatible with its differentiable structure. A *linear representation* of G is a group homomorphism $\rho : G \to GL(V)$, and the vector space V is called a *G-module.* If V is endowed with a basis, we call the image $\rho(G)$ a *matrix Lie group.*

A subspace $V_1 \subset V$ is a *G-submodule* if $\rho(G)V_1 \subseteq V_1$. A G-module is said to be *irreducible* if it has no proper G-submodules. For example, $V \otimes V$ is not irreducible as a $GL(V)$-module, since by (A.4) both $S^2 V$ and $\Lambda^2 V$ are G-submodules. On the other hand, $S^2 V$ and $\Lambda^2 V$ are irreducible.

Suppose $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are representations of G. A linear map $\alpha : V \to W$ is said to be a *G-module homomorphism* if $\alpha(\rho(g)v) = \rho'(g)\alpha(v)$ for all $v \in V$ and $g \in G$. Note that the images and kernels of G-module homomorphisms are G-submodules. If there exists a bijective G-module homomorphism between V and W, then V and W are said to be *isomorphic* G-modules, and we write $\rho \simeq \rho'$.

Lemma A.2.1 (Schur's Lemma). *Let $\rho_V : G \to GL(V)$, $\rho_W : G \to GL(W)$ be two irreducible representations of G, and let $f : V \to W$ be a linear map such that $\rho_W(g) \circ f = f \circ \rho_V(g)$ for all $g \in G$. Then:*

i. $f = 0$ unless ρ_V and ρ_W are isomorphic.

ii. If $\rho_V \simeq \rho_W$, then $f = \lambda \mathrm{Id}$.

Exercises A.2.2:

1. Prove the lemma. ⊚
2. Show that K_S, K_Λ of Exercise A.1.8.5 are isomorphic $GL(V)$-modules. The standard notation for this module is $S_{21}(V)$. ⊚

Examples. Let $Q \in S^2V^*$ be a quadratic form that is *positive definite*, i.e., $Q(v,v) > 0$ for all $v \in V\backslash 0$. We define the following subgroups of $GL(V)$:

$$\begin{aligned} SL(V) &:= \{g \in V \otimes V^* \mid \det(g) = 1\}, \\ O(V,Q) &:= \{g \in V \otimes V^* | Q(v,w) = Q(gv, gw)\ \forall v, w \in V\}, \\ SO(V,Q) &:= O(V,Q) \cap SL(V). \end{aligned}$$

These are respectively called the *special linear group*, the *orthogonal group*, and the *special orthogonal group*. We often omit reference to Q when it is understood. Note that when Q is definite, $SO(V)$ is the connected component of the identity of $O(V)$. When $V = \mathbb{R}^n$ with the standard inner product, then we write $SO(n)$ for $SO(V)$. When $V = \mathbb{R}^n$ or $\mathbb{C}^n$, we respectively write $SL(n, \mathbb{R})$ or $SL(n, \mathbb{C})$ (or SL_n when we wish to remain ambiguous) for $SL(V)$.

When $n = 2m$, a 2-form $\omega \in \Lambda^2 V^*$ is *nondegenerate* if $\omega^m = \omega \wedge \ldots \wedge \omega \neq 0$ (or, equivalently, if the map $v \mapsto v \lrcorner\, \omega$ is an isomorphism from V to V^*). In this case, ω is called a *symplectic form* on V, and we may define the *symplectic group*

$$Sp(V, \omega) := \{g \in V \otimes V^* | \omega(v \wedge w) = \omega(gv \wedge gw)\ \forall v, w \in V\}.$$

Since all nondegenerate 2-forms in $\Lambda^2(\mathbb{R}^{2m})$, are linearly equivalent, we often denote this group by $Sp(m, \mathbb{R})$ (see Exercise A.4.8.8).

Exercises A.2.3:

1. Show that all of the above are matrix Lie groups, i.e., that they are smooth submanifolds of $GL(V)$, and they are closed under multiplication and inverses.

2. Suppose $v_1, \dots, v_n$ is orthonormal with respect to Q. Determine which matrices correspond to $SL(V)$, $O(V,Q)$ and $SO(V,Q)$. For example, show that

$$SO(V) \cong SO(n) = \{A \in M_{n\times n} | {}^tAA = \mathrm{Id}\}.$$

3. Every matrix in $SO(3)$ represents a rotation fixing a line through the origin in $\mathbb{R}^3$. Is the same true for matrices in $SO(4)$?
4. A symmetric matrix S is diagonalizable by some $A \in SO(n)$, i.e., ASA^{-1} is diagonal. Show that if $S_1, \dots, S_k$ are pairwise commuting symmetric matrices, then they are simultaneously diagonalizable by an element of $SO(n)$.
5. Show that $GL(V)$ acts simply transitively on the bases of V, and thus as a manifold $GL(V)$ is isomorphic to the space of bases of V.
6. Similarly, show that $O(V)$ act simply transitively on the set of orthonormal bases of V.
7. Let V have inner product Q and let $|v| = \sqrt{Q(v,v)}$. Define $CO(V) \subset GL(V)$ to be the linear transformations preserving angles, where $\angle(v,w) := Q(v,w)/|v||w|$. Show that

$$CO(V) \cong O(V) \times \{\lambda \,\mathrm{Id} \mid \lambda > 0\}.$$

A.3. Complex vector spaces and complex structures

A *complex structure* on a (real) vector space V is a map $J \in GL(V)$ such that $J \circ J = -\mathrm{Id}$. For example, if we consider $\mathbb{C}^n = \mathbb{R}^{2n}$ as a real vector space, then multiplication by $i = \sqrt{-1}$ is not multiplication by a scalar, but it is a linear map whose square is $-\mathrm{Id}$. Define $GL(V,J) := \{g \in GL(V) | Jg = gJ\}$.

For any vector space V, we can define its *complexification* $V_{\mathbb{C}} := V \otimes \mathbb{C} = V \oplus iV$, which can be considered as the complex span of the vectors in V. So, if $\dim V = n$ then $V_{\mathbb{C}}$ has dimension n as a vector space over $\mathbb{C}$, while it has dimension $2n$ as a real vector space. (We make this distinction in notation by writing $\dim_{\mathbb{C}} V_{\mathbb{C}} = n$ and $\dim_{\mathbb{R}} V_{\mathbb{C}} = 2n$.) Any $f \in \mathrm{End}(V)$ may be extended $\mathbb{C}$-linearly to a endomorphism of $V_{\mathbb{C}}$. Note that the characteristic polynomial of this extension is the same as that of the original, so eigenvalues come in complex-conjugate pairs. The corresponding eigenvectors are also conjugate under the complex conjugation that fixes $V \subset V_{\mathbb{C}}$.

Exercises A.3.1:
1. If (V,J) is a vector space with a complex structure, show that V must be even-dimensional. ◎
2. Show that J has no eigenvectors as an endomorphism of V, but $V_{\mathbb{C}}$ splits as a direct sum of $+i$ and $-i$ eigenspaces of J. (We denote these by $V^{(1,0)}$ and $V^{(0,1)}$, respectively.)

3. Let $V = \mathbb{R}^{2m}$ with basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ and the *standard complex structure* J defined by $J(e_i) = f_i$ and $J(f_i) = -e_i$. Calculate $V^{(1,0)}$ and $V^{(0,1)}$.

4. Show that there exists a linear isomorphism $\phi : V \to \mathbb{C}^n$ such that $\phi(Jv) = i\phi(v)$ for all $v \in V$. Thus, all vector spaces with complex structures are isomorphic to the standard example.

5. Find a natural decomposition of $\Lambda^2 V_{\mathbb{C}}$ into three components invariant under the induced action of J. These components are denoted $\Lambda^{(2,0)}V$, $\Lambda^{(1,1)}V$ and $\Lambda^{(0,2)}V$. More generally, decompose $\Lambda^k V_{\mathbb{C}}$ into $k+1$ components which are J-invariant.

Unitary groups and conjugation. Let W be a complex vector space of (complex) dimension n. The invertible complex-linear endomorphisms of W form the group $GL(W) \cong GL(n, \mathbb{C})$. An *Hermitian form* h on W is a map $h : W \times W \to \mathbb{C}$ which is $\mathbb{C}$-linear in its first argument, and satisfies $h(v, w) = \overline{h(w, v)}$. Notice that $\mathsf{Re}(h)$ is a symmetric bilinear form on W considered as a real vector space. If $\mathsf{Re}(h)$ is an inner product, i.e., $\mathsf{Re}h(v, v) > 0$ for all $v \neq 0$, then h is said to be a *Hermitian inner product.* In this case we define

$$\begin{aligned} U(W, h) &:= \{g \in GL(W) |\ h(v, w) = h(gv, gw), \forall v, w \in W\} \quad &\cong U(n), \\ SU(W, h) &:= \{g \in U(W, h) | \det_{\mathbb{C}}(g) = 1\} &\cong SU(n), \end{aligned}$$

respectively the *unitary* and *special unitary groups.* Note that $\det_{\mathbb{C}}(g)$ is the determinant of the $n \times n$ matrix representing g with respect to a $\mathbb{C}$-basis.

Exercises A.3.2:

1. Relate $\det_{\mathbb{C}}(g)$ to the determinant of g when g is considered as an endomorphism of the underlying real vector space.

2. Let $h(v, w) = v \cdot \overline{w}$ for $v, w \in \mathbb{C}^n$. Show that

$$U(n) = \{U \in GL(n, \mathbb{C}) | U^{-1} = \overline{U^t}\}$$

and deduce that $|\det U| = 1$.

Given a vector space V with both an inner-product Q and a complex structure J, one can construct an Hermitian inner product whose real part is Q provided that J preserves Q, i.e., $Q(Jv, Jw) = Q(v, w)$ for all $v, w \in V$. Then $\omega(v, w) = Q(v, Jw)$ is a symplectic form on V, and we may define an Hermitian inner product by

$$h(v, w) = Q(v, w) + i\omega(v, w).$$

Exercises A.3.3:

1. Show that J preserves Q iff $J \in \mathfrak{so}(Q)$, so $J \in SO(Q) \cap \mathfrak{so}(Q)$.

2. Show that ω as defined above is also preserved by J.

Note that assuming compatibility, any two of Q, J, ω determine the third.

A.4. Lie algebras

It is usually difficult to explicitly parametrize Lie groups. Fortunately, we not need to do this, but we need to write out explicit bases for various *Lie algebras* to be defined below.

Let $\mathfrak{gl}(V) = \mathrm{End}(V) = V \otimes V^*$. After a choice of basis, we may identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices. We define a skew-symmetric multiplication $[,]$ on $\mathfrak{gl}(V)$ by

$$[X, Y] = XY - YX, \tag{A.6}$$

where XY is the usual matrix multiplication. Expanding out, one can verify the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{A.7}$$

for all $X, Y, Z \in \mathfrak{gl}(V)$.

Definition A.4.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *bracket*, that satisfies the Jacobi identity (A.7).

Exercise A.4.2: Show that $\mathbb{R}^3$ is a Lie algebra with multiplication given by the cross product.

Example A.4.3. An important class of Lie algebras is $\Gamma(TM)$, the space of smooth vector fields on a C^∞ manifold M. The bracket is $[X, Y]f := X(Yf) - Y(Xf)$, and the algebra is infinite-dimensional (see Definition B.1).

Note that one can make any vector space into a Lie algebra by using a bracket which is identically zero. Such Lie algebras are called *abelian* Lie algebras.

Inside a Lie algebra, any subspace that is closed under the bracket is called a *subalgebra*. It is a Lie algebra in its own right, since the Jacobi identity holds by restriction.

A *representation of a Lie algebra* $\mathfrak{g}$ is a linear transformation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which preserves the brackets. We say such a V is a $\mathfrak{g}$-*module*.

Example A.4.4. Let G be a Lie group. A vector field $X \in \Gamma(TG)$ is *left-invariant* if $L_{a*}(X_b) = X_{ab}$ for all $a, b \in G$, where L_{a*} denotes pushforward (see Appendix B) by left-multiplication by a. The reader can verify that the Lie bracket of two left-invariant vector fields is also left-invariant. Thus $\Gamma^L(TG)$, the space of left-invariant vector fields, is a Lie subalgebra of $\Gamma(TG)$.

A left-invariant vector field is determined by its value at just one point (say, at the identity element $e \in G$), since it is given at all other points by

pushforward under left-multiplication. Thus, we may identify $\Gamma^L(TG)$ with T_eG. We define $\mathfrak{g} = \Gamma^L(TG) \simeq T_eG$ to be the *Lie algebra of* G.

If $G \subseteq GL(V)$ is a matrix Lie group, then $\mathfrak{g} \cong T_{\mathrm{Id}}G \subset \mathfrak{gl}(V) = \mathrm{End}(V)$ is a *matrix Lie algebra.* Any Lie group G has a canonical representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by

$$\mathrm{Ad}(g)X = L_{g*}R_{g^{-1}*}X,$$

called the *adjoint representation.* For matrix Lie groups, $\mathrm{Ad}(g)X = gXg^{-1}$.

Given any representation of a Lie group G, one obtains a representation of the corresponding Lie algebra $\mathfrak{g}$ by differentiation. In particular $\mathrm{Ad} : G \to GL(\mathfrak{g})$ gives rise to a representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Exercise A.4.5: Show that $\mathrm{ad}(Y)X = [Y, X]$.

Remark A.4.6. Where does the Jacobi identity (A.7) come from? One can interpret it as a Leibniz rule. For, if A is an algebra, then a map $D : A \to A$ is a *derivation* if $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. Now take $A = \mathfrak{g}$ with the product given by the bracket and $D = D_X$ by bracket with $X \in \mathfrak{g}$.

Exercise A.4.7: Verify that $\mathrm{ad}_X : Y \mapsto [X, Y]$ is a derivation exactly because the Jacobi identity holds.

If $G \subset GL(V)$ is a matrix Lie group, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is the associated matrix Lie algebra, then $\mathfrak{g}$ has a representation on the dual spaces and tensor products constructed from V. For example, if we abbreviate $\rho(X)v$ by $X(v)$, for $X \in \mathfrak{g}$ and $v \in V$, then

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w).$$

Exercises A.4.8:
1. Let $g(t) \in G$ be a curve with $g(0) = \mathrm{Id}$ and $g'(0) = Y \in \mathfrak{g}$. Show that $\frac{d}{dt}\big|_{t=0} \mathrm{Ad}(g(t))X = [Y, X]$.
2. Consider

$$\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) \mid \operatorname{tr} X = 0\}, \tag{A.8}$$

$$\mathfrak{so}(V, Q) = \{X \in \mathfrak{gl}(V) \mid Q(Xv, w) = -Q(v, Xw)\}, \tag{A.9}$$

$$\mathfrak{gl}(V, J) = \{X \in \mathfrak{gl}(V) \mid X \circ J - J \circ X = 0\}. \tag{A.10}$$

Verify that each of these are Lie algebras, i.e., they are closed under the bracket (A.6). Show these are the translations of $T_{\mathrm{Id}}G$ to the origin for $G = SL(V), G = SO(V, Q)$ and $G = GL(V, J)$ respectively.
3. Define a group homomorphism $SL(2, \mathbb{C}) \to SO(3, \mathbb{C})$ by observing that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$ as vector spaces. What are the kernel and image of this map?

4. Show that if $X \in \mathfrak{so}(2n+1)$, then $\det(X) = 0$. Show that if $X \in \mathfrak{so}(2n)$, then the eigenvalues of X are purely imaginary and come in complex conjugate pairs and thus $\det(X)$ is nonnegative. Show that there is a smooth positive function, called the *Pfaffian* $\mathrm{Pfaff}(X)$, such that $\det(X) = \mathrm{Pfaff}(X)^2$. For those of you who like formulas, if $X = (x_{ij})$ then

$$\mathrm{Pfaff}(X) = \frac{1}{2^n n!} \sum_\sigma (\operatorname{sgn} \sigma) x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(n-1)\sigma(n)},$$

where σ runs through the permutations of $(1, \ldots, n)$

5. A quadratic form $Q \in S^2V^*$ is *nondegenerate* if the map $v \mapsto v \lrcorner Q$ is an isomorphism $V \to V^*$. Recall that Q is represented with respect to a basis $v_1, \ldots, v_n$ by a symmetric matrix with entries $Q_{ij} = Q(v_i, v_j)$. We say Q has *signature* (p, q) if its matrix has p positive eigenvalues and q negative eigenvalues. If Q is nondegenerate, show that there is a basis such that

$$Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix.

Show that if Q has signature $(n, 0)$, and we choose bases such that $Q = \mathrm{Id}_n$, then $\mathfrak{so}(V, Q) \cong \{X | X + X^t = 0\}$.

6. If Q has signature $(3, 1)$, we could take bases such that

$$Q = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we use the first basis, we have

$$\mathfrak{so}(3,1) \simeq \left\{ \left. \begin{pmatrix} 0 & x_1^2 & x_1^3 & x_1^4 \\ x_1^2 & 0 & -x_2^3 & -x_2^4 \\ x_1^3 & x_2^3 & 0 & -x_3^4 \\ x_1^4 & x_2^4 & x_3^4 & 0 \end{pmatrix} \right| x_1^2, x_1^3, \ldots \in \mathbb{R} \right\}.$$

Find the form of $\mathfrak{so}(3, 1)$ using the second choice for Q.

7. Similarly, given a complex structure on $\mathbb{R}^4$, two choices of bases yield

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Find the form of $\mathfrak{gl}(2, \mathbb{C}) \subset \mathfrak{gl}(4, \mathbb{R})$ using these two choices for J.

8. Two choices of symplectic form on $V \simeq \mathbb{R}^{2n}$ are

$$\omega = dx^1 \wedge dx^{n+1} + \ldots + dx^n \wedge dx^{2n},$$
$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2n-1} \wedge dx^{2n}.$$

Find the two corresponding forms of $\mathfrak{sp}(n,\mathbb{R})$.

9. The symmetric bilinear form $B(X,Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ on a Lie algebra is called the *Killing form*. Compute the Killing forms for $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{gl}(2,\mathbb{C})$ and $\mathfrak{sp}(2,\mathbb{R})$.

Example A.4.9. Taking $e_1, e_2 \in \mathbb{C}^2$ as a unitary basis gives

$$\mathfrak{su}(2) = \left\{ \frac{1}{2} \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\} \subset \mathfrak{gl}(2,\mathbb{C}).$$

Let $S^4(\mathbb{C}^2)$ have basis $v_1 = e_1^4$, $v_2 = 2e_1^3 e_2$, $v_3 = \sqrt{6}e_1^2 e_2^2$, $v_4 = 2e_1 e_2^3$, $v_5 = e_2^4$. Then the induced representation of $\mathfrak{gl}(2,\mathbb{C})$ on $S^4(\mathbb{C}^2)$ restricts to give the following representation of $\mathfrak{su}(2)$:

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} \\ &\mapsto \begin{pmatrix} 2ix & -y+iz & 0 & 0 & 0 \\ y+iz & ix & \sqrt{\frac{3}{2}}(-y+iz) & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}(y+iz) & 0 & \sqrt{\frac{3}{2}}(-y+iz) & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}(y+iz) & -ix & -y+iz \\ 0 & 0 & 0 & y+iz & -2ix \end{pmatrix}. \end{aligned}$$

Exercise A.4.10: Take an orthonormal basis for $\mathbb{R}^3$, and find an orthonormal basis of $V = S^2(\mathbb{R}^3)$. Find the representation $\mathfrak{so}(3) \to \mathfrak{gl}(V)$ resulting from the induced representation of $\mathfrak{gl}(3,\mathbb{R})$.

A.5. Division algebras and the simple group G_2

In addition to the orthogonal and symplectic groups, there is just one more class of groups that can be defined as the group preserving a generic tensor of some type on V. The reason is that the tensor spaces have dimension greater than that of $GL(V)$—for example, S^3V has dimension $\frac{(n+2)(n+1)}{6} > n^2$, where $n = \dim V$—so the subgroup $G \subset GL(V)$ preserving a generic element must be zero-dimensional.

In fact, the only potential examples are $\Lambda^3 V$ for dimensions $n = 6, 7$ or 8. We expect the corresponding groups to have dimension 16, 14 and 8 respectively.

In the case $n = 8$, note that $V = \mathfrak{sl}(3)$ (over either $\mathbb{R}$ or $\mathbb{C}$) has dimension eight and there is a natural 3-form induced by the bracket $[,] : V \times V \to V$. Identifying V with V^* via the Killing form, we obtain an element $\phi \in V^{*\otimes 3}$.

Exercise A.5.1: Show that $\phi \in \Lambda^3 V^*$.

By its invariant definition, ϕ is preserved by $SL(3)$, and a little more work shows that ϕ is generic, so $\mathfrak{sl}(3)$ coincides with $\mathrm{der}(\phi)$, the Lie algebra of $\mathrm{Aut}(\phi) = G$.

In the case $n = 7$, first note that a generic $\phi \in \Lambda^3 V^*$ and a volume form $\Omega \in \Lambda^7 V^*$ determine a bilinear form $B(v, w)$ defined by

$$(v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi = B(v, w)\Omega.$$

Exercise A.5.2: Show that B is symmetric and nondegenerate, and thus $G(\phi) \subseteq CO(B, V)$. (We obtain the conformal group, as B is a priori only well-defined up to the choice of volume form.)

Over the reals, two types of signature are possible, one of which is definite. Say we are in the case where the signature is definite; then, by comparing homogeneity, one can see that B is in fact well-defined. A little more calculation (see [**74**], Theorem 6.80) shows that one obtains a new simple Lie group of dimension 14, which is named G_2.

G_2 and the octonions. The compact form of the group G_2, corresponding to B definite, is also the automorphism group of the *octonions*. (In fact, its presentation as above was only discovered relatively recently by Bryant [**74**].)

Recall that the set of quaternions $\mathbb{H}$ is a normed division algebra that is a four-dimensional vector space over $\mathbb{R}$. Elements of $\mathbb{H}$ may be written as $x = x^0 + x^1 e_1 + x^2 e_2 + x^3 e_3$ where $x^j \in \mathbb{R}$ and the symbols e_j satisfy the multiplication rule ${e_j}^2 = -1$ and either of the following equivalent multiplication tables:

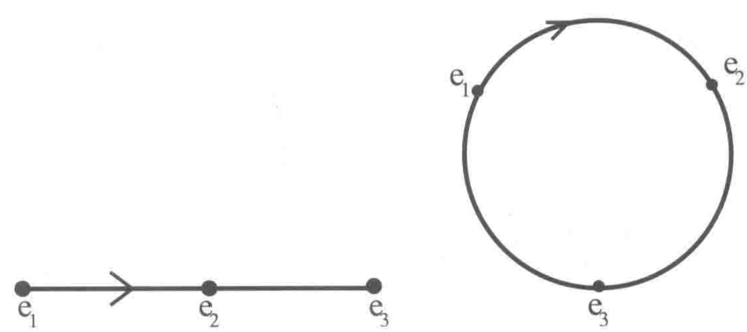


Figure 1. Products are positive if one multiplies with the arrow (e.g., $e_1 e_2 = e_3$), negative against (e.g. $e_3 e_2 = -e_1$)

The octonions (also known as *Cayley numbers*) form an eight-dimensional vector space over $\mathbb{R}$, in which elements may be written as $x = x^0 + x^1 e_1 + \ldots + x^7 e_7$ with $x^j \in \mathbb{R}$ and the symbols e_j satisfying ${e_j}^2 = -1$ and the multiplication rules in Figure 2.

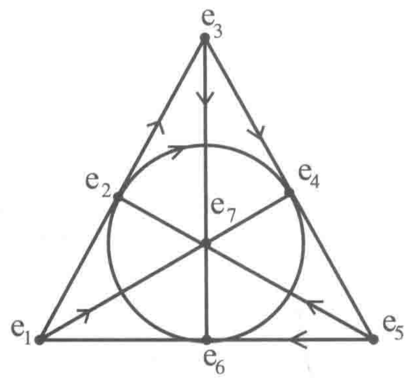


Figure 2. Multiplication rules for the basis for $\mathsf{Im}\mathbb{O}$

Taking $V = \mathsf{Im}\mathbb{O} = \{e_1, \ldots, e_7\}$, we may define the form ϕ in terms of octonionic multiplication:

$$\phi(x, y, z) = -\tfrac{1}{2}\mathsf{Re}[x(yz) - z(yx)], \qquad x, y, z \in \mathsf{Im}\mathbb{O}. \tag{A.11}$$

In fact, there are only four division algebras over $\mathbb{R}$: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. Suppose $\mathbb{A}$ is one of these, and let

$$\mathrm{Aut}(\mathbb{A}) := \{g \in GL(\mathbb{A}) \mid (gu)(gv) = g(uv)\, \forall u, v \in \mathbb{A}\}.$$

Then $\mathrm{Aut}(\mathbb{A})$ is respectively $\{\mathrm{Id}\}$, $\mathbb{Z}_2$, $SL(2, \mathbb{C})$, or G_2.

The split form of G_2, corresponding to B indefinite, is the automorphism group of the split octonions [**74**], and also occurs as the automorphism group of a very important Pfaffian system on a 5-manifold, see [**27**]. (The action on the 5-manifold is not linear; the lowest-dimensional linear representation of G_2 has dimension seven.)

Exercises A.5.3:

1. Prove the *Moufang identities*: for $a, b, c \in \mathbb{O}$,

$$\begin{aligned} ((ab)a)c &= a(b(ac)), \\ c(a(ba)) &= ((ca)b)a, \\ (ab)(ca) &= a(bc)a. \end{aligned} \tag{A.12}$$

2. Use the Moufang identities to verify that $\phi(x, y, z)$ is skew-symmetric in x, y, z.
3. Similarly, verify that the *associator*

$$[a, b, c] := \tfrac{1}{2}[(ab)c - a(bc)], \qquad a, b, c \in \mathbb{O},$$

is skew-symmetric in a, b, c.
4. Calculate ϕ in terms of coordinates $x^1, \ldots, x^7$ on $\mathbb{R}^7 = \mathsf{Im}\mathbb{O}$. ◎

5. Show that the Lie algebra $\mathfrak{g}_2 = \mathrm{der}(\phi)$ has the form

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} \kappa & -{}^tA \\ A & \rho(\kappa \oplus \tau) \end{pmatrix} \middle| \, \kappa, \tau \in \mathfrak{so}(3) \right\}, \tag{A.13}$$

where ρ is the isomorphism $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{so}(4)$ and $a_{13} = a_{31} + a_{42}$, $a_{23} = a_{41} - a_{32}$, $a_{33} = a_{22} - a_{11}$, and $a_{43} = -a_{21} - a_{12}$.
6. Determine the classical group(s) preserving a generic $\phi \in \Lambda^3\mathbb{R}^6$.

A.6. A smidgen of representation theory

This section contains a brief overview of the rudiments of representation theory. For proofs and more complete statements of what follows, see any of the standard texts such as [**15, 52, 80, 93**].

Why representation theory? In general, when a group G (or Lie algebra $\mathfrak{g}$) acts on a vector space V, one would like to know the decomposition of V into irreducible G-modules, if such a decomposition exists. The utility of having such decompositions will become obvious as you read through this book. For now, consider the following motivating examples:

Example A.6.1. Let (M^n, g) be a Riemannian manifold. Say $c : M \to \mathbb{R}_+$ is a smooth function and consider the new Riemannian metric cg. How does the curvature tensor (see Chapter 2) change under such a change of metric? Is any aspect of it unchanged? Since the group of rotations which preserve the metric pointwise is also unchanged, one might suspect that the answer involves the action of these rotations on the curvature.

Representation theory helps us split up the curvature tensor into irreducible pieces and see how each piece changes. In particular, we'll see there is a piece that doesn't change, first discovered by H. Weyl when he was investigating Einstein's theory of general relativity. Weyl's study was what motivated him to make his fundamental contributions to representation theory (see [**76**] for more detail).

Example A.6.2. Consider the rational normal curve $C_d \subset \mathbb{P}^d = \mathbb{P}(S^d(\mathbb{C}^2))$, which is the image of $\mathbb{CP}^1$ under $[x, y] \mapsto [x^d, x^{d-1}y, x^{d-2}y^2, \dots, y^d]$, or, without coordinates, $[v] \mapsto [v \circ v \circ \dots \circ v]$ for $v \in \mathbb{C}^2\backslash 0$. (see Chapter 3). What are the linear changes of coordinates in $\mathbb{C}^{d+1}$ that leave C_d invariant? The set of all such changes forms a subgroup of $GL(d + 1, \mathbb{C})$ which is isomorphic to $GL(2, \mathbb{C})$. The action may be seen explicitly by $g.[v_1 \circ \dots \circ v_d] = [(gv_1) \circ \dots \circ (gv_d)]$.

Exercise A.6.3: Show that the representation $\rho : GL(2, \mathbb{C}) \to GL(d + 1)$ described above is irreducible. ◎

We will outline how to understand representations in one easy case, when the Lie algebra of G is *simple*. A Lie algebra $\mathfrak{g}$ (or a Lie group G) is

called *reductive* if all $\mathfrak{g}$-modules decompose into a direct sum of irreducible $\mathfrak{g}$-modules, *simple* if it has no nontrivial ideals, and *semi-simple* if it is the direct sum of simple Lie algebras. Semi-simple Lie algebras are reductive, but not all reductive Lie algebras are semi-simple. For example, $\mathfrak{sl}(V)$ is simple while $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \{\lambda \operatorname{Id}\}$ is reductive. However, it turns out that all reductive Lie algebras are either semi-simple or semi-simple plus an abelian algebra.

We discuss irreducible representations of simple Lie algebras. The irreducible representations of a semi-simple Lie algebra are just the tensor products of the irreducible representations of its simple components. Thanks to the action of $\mathfrak{g}$ on itself, without loss of generality, we may assume that $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a matrix Lie algebra.

Simple Lie algebras are best studied via a certain kind of abelian subalgebra they contain. We decompose a given $\mathfrak{g}$-module V first with respect to the matrices in such a subalgebra.

Exercise A.6.4: Let $A_1, \dots, A_r$ be $n \times n$ matrices that commute. Show that if each A_j is diagonalizable, then $A_1, \dots, A_r$ are simultaneously diagonalizable. ◎

If a matrix A is diagonalizable, then V decomposes into eigenspaces for A and there is an eigenvalue associated to each eigenspace. Now let $\mathfrak{t} = \{A_1, \dots, A_r\} \subset \mathfrak{g}$ be the subspace spanned by $A_1, \dots, A_r$ as above. Then V decomposes into simultaneous eigenspaces for all $A \in \mathfrak{t}$. For each eigenspace V_j define a function $\lambda_j : \mathfrak{t} \to \mathbb{R}$ such that $\lambda_j(A)$ is the eigenvalue of A associated to the eigenspace V_j. Note that λ_j is a *linear* map, so we may think of $\lambda_j \in \mathfrak{t}^*$.

If there are p distinct eigenspaces of V, then these λ_j give p elements of $\mathfrak{t}^*$ which are called the *weights* of V. The dimension of V_j is called the *multiplicity* of λ_j in V. The decomposition $V = \bigoplus_j V_j$ is called the *weight space decomposition* of V.

Exercise A.6.5: Show that the only irreducible representations of an abelian Lie algebra $\mathfrak{t}$ are one-dimensional. ◎

Now, back to our simple Lie algebra $\mathfrak{g}$. There always exists a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$, unique up to conjugation by G, called a *maximal torus.* We define the *rank* of $\mathfrak{g}$ to be the dimension of a maximal torus. Amazingly, representations of $\mathfrak{g}$ are completely determined up to equivalence by how $\mathfrak{t}$ acts. More precisely, suppose V is an irreducible $\mathfrak{g}$-module. Then as a $\mathfrak{t}$-module V admits a weight space decomposition. If two $\mathfrak{g}$ modules V, W have the same weights (with the same multiplicities) as $\mathfrak{t}$-modules, then they are isomorphic as $\mathfrak{g}$-modules.

Let $\mathfrak{g}$ act on itself by the adjoint action. Then as a $\mathfrak{t}$-module, we have the weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R \subset \mathfrak{t}^*$ is some finite subset whose nonzero members are called the *roots* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is the eigenspace associated to each root. (Of course, $\mathfrak{g}_0 = \mathfrak{t}$ since $\mathfrak{t}$ is maximal.) Another amazing fact is that the eigenspaces $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ are all one-dimensional.

Remark A.6.6 (Why the word "root"?). Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ restricted to $\mathfrak{t}$. The roots of the characteristic polynomial $p_\lambda(X) = \det(\mathrm{ad}(X) - \lambda \,\mathrm{Id}_{\mathfrak{g}})$ are the eigenvalues of X. By varying X one obtains linear functions on $\mathfrak{t}$ which are the roots of $\mathfrak{g}$.

Exercise A.6.7: In fact, weights were originally called "generalized roots" when the theory was being developed. They too are roots of a characteristic polynomial—which one?

Let's look at some of our favorite simple Lie algebras:

Example A.6.8 ($\mathfrak{g} = \mathfrak{sl}(n)$). Here the rank of $\mathfrak{g}$ is $n-1$, and we may take

$$\mathfrak{t}(x^1, \ldots, x^{n-1}) = \begin{pmatrix} x^1 & & & \\ & \ddots & & \\ & & x^{n-1} & \\ & & & -(x^1 + \ldots + x^{n-1}) \end{pmatrix}.$$

Let $e^i_j = v^i \otimes v_j$. Then we have $\mathfrak{t}(e^i_j) = (x^i - x^j)e^i_j$ for $i \neq j$. So the roots are $x^i - x^j$ and $\mathfrak{g}_{x^i - x^j} = \{e^i_j\}$.

Example A.6.9 ($\mathfrak{g} = \mathfrak{so}(2n)$). The rank is n, and we may take

$$\mathfrak{t}(x^1, \ldots, x^n) = \begin{pmatrix} 0 & x^1 & & & \\ -x^1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x^n \\ & & & -x^n & 0 \end{pmatrix}.$$

Exercise A.6.10: Determine the roots and the root space decomposition for $\mathfrak{so}(2n)$.

Remark A.6.11. The classification of simple Lie algebras $\mathfrak{g}$ (due to Killing and Cartan) is based on classifying the possible *root systems*, the collection of roots for $\mathfrak{g}$. The rules for such systems are rather strict and concise. For the record, we need a subset $R \subset \mathfrak{t}^*$ such that

(1) R must span $\mathfrak{t}^*$ and moreover if $\mathfrak{g}^*$ is complex, R is an $\mathbb{R}$-linear (in fact $\mathbb{Q}$-linear) subspace;

(2) for each $\alpha \in R$, reflection in the hyperplane perpendicular to α must map R to R;

(3) for all $\alpha, \beta \in R$, the quantity $2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle$ must be an integer; and

(4) for all $\alpha \in R$, $2\alpha \notin R$.

(The inner product $\langle\, ,\rangle$ on $\mathfrak{t}$ is minus one times the restriction of the Killing form.) When $\mathfrak{g}$ is simple, R also cannot be split up into two separate root systems in complementary subspaces.

If $\mathfrak{g}$ is a simple algebra, then to each irreducible representation of $\mathfrak{g}$ is associated a set of weights (points in $\mathfrak{t}^*$), each with some multiplicity. Not every set of points in $\mathfrak{t}^*$ corresponds to an irreducible representation. In the first place, the admissible points lie on a lattice, called the *weight lattice*. The weight lattice is the set of $\ell \in \mathfrak{t}^*$ such that $\langle \ell, \alpha'\rangle \in \mathbb{Z}$ for all $\alpha' \in L_{R'}$, where $L_{R'}$ is the lattice generated by the *co-roots* $\alpha' = \frac{2}{\langle\alpha,\alpha\rangle}\alpha$ for $\alpha \in R$.

Moreover, the weights that arise in a given irreducible representation are uniquely determined once one fixes a *highest weight*. That is, an appropriate order is placed on the weight lattice, and the extremal weight of an irreducible representation (which necessarily has multiplicity one) determines all other weights, along with their multiplicities. (In fact, the other weights are obtained by translating the highest weight by the negative roots.) A vector in V is called a *weight vector* if it is in an eigenspace for the torus and a *highest weight vector* if the corresponding weight $\mathfrak{t}^*$ determining the eigenvalues is a highest weight.

The weight lattice has $r = \dim \mathfrak{t}^*$ generators, so once one fixes a set of generators $\omega_1, \ldots, \omega_r$, the irreducible representations correspond to r-tuples of non-negative integers $(\ell_1, \ldots, \ell_r)$ which determine a highest weight $\ell = \ell_1\omega_1 + \ldots + \ell_r\omega_r$.

Example A.6.12. When $\mathfrak{g} = \mathfrak{sl}(n) = \mathfrak{sl}(V)$, the weight lattice is generated by $x^1, x^1 + x^2, \ldots, x^1 + \ldots + x^{n-1}$. These are the highest weights of the irreducible representations $V, \Lambda^2 V, \ldots, \Lambda^{n-1} V$ respectively.

Exercise A.6.13: Verify the last statement by computing the action of $\mathfrak{t}$ on the highest weight vector $e_1 \wedge \ldots \wedge e_k$ of $\Lambda^k V$.

Remark A.6.14. Even when G is not simple, studying weights of a representation (i.e., weights under the maximal torus of a maximal semi-simple subgroup of G) can yield useful information; see §8.6.

Some remarks useful for Riemannian geometry.

Exercise A.6.15: Consider the skew-symmetrization map $\delta : \Lambda^2 V \otimes \Lambda^2 V \to \Lambda^4 V$ defined by $\delta(\alpha, \beta) = \frac{1}{4}\mathcal{A}(\alpha \otimes \beta)$, where $\mathcal{A}$ is defined by (A.3). Let δ' denote its restriction to $S^2(\Lambda^2 V) \subset \Lambda^2 V \otimes \Lambda^2 V$. Show that both δ and δ' are $GL(V)$-module homomorphisms.

Let $S_{22}(V) := \ker \delta'$. Show that this space has dimension $\frac{n^2(n^2-1)}{12}$.

The $GL(V)$-module $S_{22}(V)$, which is often denoted $\mathcal{K}(V)$, plays an important role in Riemannian geometry, since it is the space of possible Riemann curvature tensors. Let $V = T_x^* M$, where x is an arbitrary point in a Riemannian manifold (M, g). If $R \in \mathcal{K}(V)$ is the curvature, we may think of R as a quadratic polynomial on $\Lambda^2 T_x M$, which restricts to a function on the Grassmannian $G(2, T_x M)$. If $E \in G(2, V)$, then $R(E)$ is called the *sectional curvature* of the plane $E \subset T_x M$.

Exercises A.6.16:

1. Show that, as an $SO(V)$-module, $\mathcal{K}(V)$ decomposes into three components, denoted $\mathrm{Scal}(V)$, $\mathrm{Ric}_0(V)$ and $\mathrm{Weyl}(V)$. The corresponding components of the curvature tensor are the *scalar curvature*, the *traceless Ricci curvature*, and the *Weyl curvature* respectively. As $SO(V)$-modules, $\mathrm{Ric}(V) = \mathrm{Ric}_0(V) + \mathrm{Scal}(V) \simeq S^2 V$ and $\mathrm{Scal}(V) \simeq \mathbb{R}$. In answer to our question posed in the motivation, $\mathrm{Weyl}(V)$ is unchanged under conformal changes of metric. Prove this by considering how $\lambda\,\mathrm{Id}$ acts on Weyl. ◎

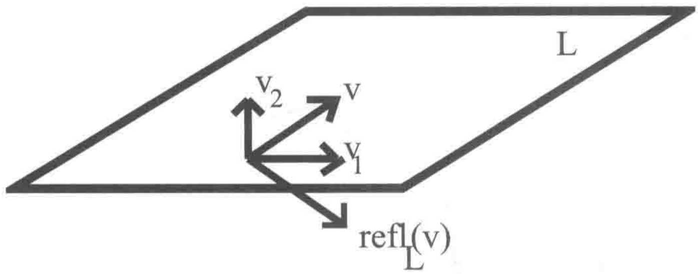
2. Consider the map $\delta' : S^2(\Lambda^2 V) \otimes V \to \Lambda^2 V \otimes \Lambda^3 V$, given by extending the wedge product $\Lambda^2 V \otimes V \to \Lambda^3 V$ to $\Lambda^2 V \otimes \Lambda^2 V \otimes V \to \Lambda^2 V \otimes \Lambda^3 V$, and then restricting. Restrict δ' to $\mathcal{K}(V) \otimes V$ and define $\mathcal{K}^{(1)}(V) = S_{32}(V) = \ker \delta'$. Show that there is an exact sequence of vector spaces
$$0 \to \mathcal{K}^{(1)}(V) \to \mathcal{K}(V) \otimes V \to \Lambda^2 V \otimes \Lambda^3 V \to V \otimes \Lambda^4 V \to 0,$$
and thus
$$\dim \mathcal{K}^{(1)}(V) = \frac{n^2(n^2-1)(n+2)}{24}.$$

3. Consider the mapping
$$\gamma : W \otimes S^2 V \to S^2(\Lambda^2 V),$$
$$h^a_{ij} w^a v^i v^j \mapsto \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}) v^i \otimes v^j \otimes v^k \otimes v^l.$$
Show that $\gamma(W \otimes S^2 V) \subseteq \mathcal{K}(V)$.

A.7. Clifford algebras and spin groups

Let V be a complex vector space with a nondegenerate quadratic form $Q \in S^2 V^*$. Given any linear subspace $L \subset V$, one can define its Q-orthogonal complement $L^{\perp Q} = \{w \in V \mid Q(v, w) = 0\, \forall v \in L\}$. If $Q|_L$ is nondegenerate,

then $V = L \oplus L^{\perp Q}$. In that case, for all $v \in V$ we may write $v = v_1 + v_2$ for $v_1 \in L$, $v_2 \in L^{\perp Q}$, and we define the *reflection* of v in L by $\mathrm{refl}_L(v) = v_1 - v_2$.



Recall that $O(V,Q)$ is the subgroup of $GL(V)$ preserving Q, and $SO(V,Q) = O(V,Q) \cap SL(V)$.

Theorem A.7.1 (Cartan-Dieudonné [**74**]). *The group $O(V,Q)$ is generated by reflections in lines, and $SO(V,Q)$ is generated by compositions of even numbers of reflections. More precisely, $O(V,Q) = \{\mathrm{refl}_{l_1} \circ \ldots \circ \mathrm{refl}_{l_k} \mid l_j \in \mathbb{P}V\}$ where we may assume $k \leq n$, and similarly for $SO(V,Q)$ except k must be even.*

To define $\mathrm{Spin}(V,Q)$, the connected and simply-connected group with Lie algebra $\mathfrak{so}(V,Q)$, we will need to generalize the notion of a reflection. Let $\Lambda^\bullet V = V^{\otimes}/\langle x \otimes y + y \otimes x\rangle$ be the exterior algebra on V, where $\langle x \otimes y + y \otimes x\rangle$ denotes the ideal generated by expressions of the form $x \otimes y + y \otimes x$ with $x, y \in V$. Note that Q induces a quadratic form on $\Lambda^\bullet V$ which we also denote by Q. The *exterior product* in $\Lambda^\bullet V$, $(x,y) \mapsto x \wedge y$, may be interpreted as follows: Let $\hat{G}(i,V) \subset \Lambda^i V$ denote the cone over the Grassmannian. If $x \in \hat{G}(i,V)$, $y \in \hat{G}(j,V)$, then $x \wedge y \in \hat{G}(i+j,V) \subset \Lambda^{i+j} V$ represents the $i+j$-plane spanned by x and y. If $x \in V$, then $x \wedge y$ is analogous to the component of x in $y^{\perp Q}$. If $||y||_Q = 1$, then $||x \wedge y||_Q = ||\,\mathrm{proj}_{y^{\perp Q}}(x)||_Q$. Note that we do not need Q to define $x \wedge y$.

Let $x \lrcorner\, y$ be defined to be the Q-adjoint of $x \wedge y$, that is, $Q(x \lrcorner\, y, z) = Q(y, x \wedge z)$ for all $x, y, z \in \Lambda^\bullet V$. For example, if $x, y \in V$, then $x \lrcorner\, y = Q(x,y)$. If $x \in V$ and $y \in \hat{G}(j,V)$, then $x \lrcorner\, y$ is analogous to the component of x in y. Note that if y has unit length, then $||x \lrcorner\, y||_Q = ||\,\mathrm{proj}_y(x)||_Q$.

Exercise A.7.2: Show that $x \lrcorner\, (x \lrcorner\, y) = 0$ for all $x, y \in \Lambda^\bullet V$.

Consider, for $x, y \in \Lambda^\bullet V$,

$$x \circ y := x \wedge y - x \lrcorner\, y,$$

which can be thought of as the "generalized reflection" of x in y.

Definition A.7.3. Let V be a vector space with a quadratic form Q. Let $Cl(V,Q) := (\Lambda^\bullet V, \circ)$, the *Clifford algebra* of (V,Q).

Exercise A.7.4: Show that
$$Cl(V,Q) = V^{\otimes}/\langle x \otimes y + y \otimes x - 2Q(x,y)\rangle.$$

Lemma A.7.5 (Fundamental Lemma of Clifford algebras). *Let V be a vector space with a quadratic form Q and let $\mathcal{A}$ be an associative algebra with unit. If $\phi : V \to \mathcal{A}$ is a mapping such that for all $x, y \in V$*
$$\phi(x)\phi(y) + \phi(y)\phi(x) = 2Q(x,y)\,\mathrm{Id}_{\mathcal{A}},$$
then ϕ has a unique extension to an algebra mapping $\tilde{\phi} : Cl(V,Q) \to \mathcal{A}$.

For a proof, see [**74**].

Exercise A.7.6: Show that equivalently it is sufficient that ϕ satisfies $\phi(x)^2 = 2||x||_Q^2\,\mathrm{Id}_{\mathcal{A}}$ for all $x \in V$.

In $Cl(V,Q) = (\Lambda^\bullet V, \circ)$, the degree of a form is no longer well-defined, but there is still a notion of parity. Let $Cl^{even}(V,Q), Cl^{odd}(V,Q) \subset Cl(V,Q)$ denote the corresponding even and odd subspaces. As vector spaces, $Cl^{even}(V,Q) = \Lambda^{even}V$, $Cl^{odd}(V,Q) = \Lambda^{odd}V$.

Exercise A.7.7: Verify that the parity is well-defined.

Definition A.7.8. Let $Cl^*(V,Q) \subset Cl(V,Q)$ denote the invertible elements. Let
$$\mathrm{Pin}(V,Q) := \{a \in Cl^*(V,Q) \mid a = u_1 \circ \ldots \circ u_r, u_j \in V, Q(u_j,u_j) = 1\},$$
$$\mathrm{Spin}(V,Q) := \{a \in \mathrm{Pin}(V,Q) \mid r \text{ is even }\}.$$

Exercises A.7.9:

Given $a = u_1 \circ \ldots \circ u_r \in Cl(V,Q)$, let $\tilde{a} = (-1)^r u_r \circ \ldots \circ u_1$.

1. $a \mapsto \tilde{a}$ is a well-defined involution.
2. If $v \in V$, then $a \circ v \circ \tilde{a} \in V$.
3. Using the tilde involution, we define a representation $\rho : \mathrm{Spin}(V,Q) \to GL(V)$ by $\rho(a)v := av\tilde{a}$. Show that ρ is a 2-to-1 map $\mathrm{Spin}(V,Q) \to SO(V,Q)$. ◎

Clifford algebras as matrix algebras. From now on, assume $\dim V = 2m$.

We have defined $Cl(V,Q)$ as $\Lambda^\bullet V$ with an exotic multiplication, but in fact, as an algebra, $Cl(V,Q)$ is something familiar, as we now show. Fix $U, U' \subset V$ such that $Q|_U = Q|_{U'} = 0$ and $V = U + U'$. (Note that this implies $\dim U = \dim U'$, $U \cap U' = 0$ and $V = U \oplus U'$.) Thus for all $v \in V$ we may uniquely write $v = x + y$ with $x \in U$, $y \in U'$. Define a mapping
$$\phi : V \to \mathrm{End}(\Lambda^\bullet U),$$
$$v = x + y \mapsto (u \mapsto \sqrt{2}(x \wedge u - y \lrcorner u)).$$

We calculate

$$\begin{aligned}\phi(v)^2 u &= 2(x \wedge (x \wedge u - y \lrcorner u) - y \lrcorner (x \wedge u - y \lrcorner u)) \\ &= x \wedge x \wedge u - x \wedge (y \lrcorner u) - y \lrcorner (x \wedge u) - y \lrcorner (y \lrcorner u) \\ &= 2Q(x, y)u = ||v||_Q^2 u.\end{aligned}$$

Thus the fundamental lemma applies and we obtain an algebra map $\tilde{\phi} : Cl(V, Q) \to \mathrm{End}(\Lambda^\bullet U)$.

Exercises A.7.10:

1. $\tilde{\phi}$ is a bijection, and thus, as an algebra, $Cl(V, Q) \simeq \mathrm{End}(\Lambda^\bullet U)$.
2. Moreover,

$$\begin{aligned}Cl^{even}(V, Q) &\simeq \mathrm{End}(\Lambda^{even} U) \oplus \mathrm{End}(\Lambda^{odd} U), \\ Cl^{odd}(V, Q) &\simeq (\Lambda^{even} U)^* \otimes (\Lambda^{odd} U) \oplus (\Lambda^{odd} U)^* \otimes (\Lambda^{even} U).\end{aligned}$$

3. Finally, show that $\mathrm{Spin}(V, Q)$ preserves $\mathrm{End}(\Lambda^{even} U)$. Standard notation is

$$\begin{aligned}\mathrm{End}(\Lambda^{even} U) &= \mathcal{S}_+, \\ \mathrm{End}(\Lambda^{odd} U) &= \mathcal{S}_-,\end{aligned}$$

the space of *positive* (resp. *negative*) *spinors*.

4. If $\dim V > 7$, show that $Cl(V)$ cannot act nontrivially on $\mathbb{C}^n$ for $n < 8$. If $\dim V \leq 7$, determine the values of n such that $Cl(V)$ acts nontrivially on $\mathbb{C}^n$.

Differential Forms

B.1. Differential forms and vector fields

Let M^n be a differentiable manifold, let $C^\infty(M)$ denote the set of smooth real-valued functions on M, and let $x \in M$. The *cotangent space* of M at x, denoted T_x^*M, may be defined as the set of equivalence classes of maps $f \in C^\infty(M)$, where $f \sim g$ if in any coordinate system their first derivatives agree at x. We let df_x denote the equivalence class of f. The cotangent space can also be defined as the quotient of the maximal ideal $\mathfrak{m}_x$ of functions vanishing at x by the ideal $\mathfrak{m}_x^2$ of functions vanishing to order two, i.e., $T_x^*X = \mathfrak{m}_x/\mathfrak{m}_x^2$.

Notation B.1.1. If $\pi : E \to M$ is a vector bundle, we let $\Gamma(E)$ denote the space of smooth *sections* of E, i.e., C^∞ maps $s : M \to E$ such that $\pi \circ s = \mathrm{Id}$.

Exercises B.1.2:

1. Show that T_x^*M is a vector space.
2. Show that if x^i are local coordinates on M, then the $dx^i|_x$ provide a basis of T_x^*M, and thus $\dim T_x^*M = \dim M$.
3. Show that the *cotangent bundle* $T^*M = \bigcup_{x\in M} T_xM$ is a vector bundle over M. Thus, given $f : M \to \mathbb{R}$, we obtain a section $df \in \Gamma(T^*M)$.
4. We define T_xM, the *tangent space* of M at x, as the space of equivalence classes of differentiable mappings $f : \mathbb{R} \to M$ with $f(0) = x$, where $f \sim g$ if their first derivatives agree in a coordinate system. Show that if $f \sim g$, then their derivatives agree in every coordinate system. The vector bundle $TM = \bigcup_{x\in M} T_xM$ is called the *tangent bundle*, and sections $X \in \Gamma(TM)$ are called *vector fields*.

5. Show that T_xM and T_x^*M are naturally dual vector spaces. (Hint: Consider the composition of mappings.)

The pairing between T_xM and T_x^*M allows us to differentiate a function f using a vector field X. Namely, for $X \in \Gamma(TM)$ and a function $f : M \to \mathbb{R}$, we define

$$X(f)_x = \langle df_x, X_x \rangle.$$

Note that a vector field is determined by how it acts on functions.

Differentiating a smooth function by a vector field produces another function, which we may differentiate again. The failure of successive differentiation by vector fields X and Y to commute is measured by their *Lie bracket* $[X, Y]$, which is a vector field determined by

$$[X, Y](f) := X(Y(f)) - Y(X(f)). \tag{B.1}$$

Exercises B.1.3:
1. Suppose $(x^1, \ldots, x^n)$ are local coordinates[1] and $X = a^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}$. Compute $X(f)$ in these coordinates, and conclude that the value of $X(f)_x$ depends only on the value of X_x.
2. If, in addition, $Y = b^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}$, compute $[X, Y]$.

Notation B.1.4. The pairing between sections of TM and T^*M produces a function on M which may be denoted in several ways:

$$\langle \alpha, X \rangle = \alpha(X) = X \lrcorner\, \alpha, \qquad \alpha \in \Gamma(T^*M),\ X \in \Gamma(TM).$$

Let $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$. The direct sum $\Omega^*(M) := \bigoplus \Omega^k(M)$ denotes the space of all differential forms on M.

If $\alpha \in \Omega^k(M)$, we call α a *differential form of degree k*, or *k-form* for short. As indicated in Exercise B.1.2.2, any 1-form can be expressed as a linear combination of the dx^i's, where $x^1, \ldots, x^n$ are local coordinates on M. In turn, any k-form can be expressed as a linear combination of wedge products of 1-forms (see Appendix A).

Exercise B.1.5: On the circle in $\mathbb{R}^2$ defined by $x^2 + y^2 = 1$, let $\theta \in (0, 2\pi)$ be the angle from the positive x-axis and $\psi \in (0, 2\pi)$ be the angle from the negative x-axis, both measured counterclockwise. Show that $d\theta = d\psi$ where both angles are defined. Thus, one can define $\alpha \in \Omega^1(S^1)$ by $\alpha = d\theta$ where θ is defined, and otherwise $\alpha = d\psi$.

Show that $\alpha \neq df$ for any function $f : S^1 \to \mathbb{R}$. Thus $\{df \mid f \in C^\infty(M, \mathbb{R})\} \subsetneq \Omega^1(M)$ when $M = S^1$.

[1] When expressing vector fields and differential forms in terms of a basis, we again use the convention that summation is implied over repeated indices.

Pullbacks of differential forms, pushforwards of vector fields. Let $\phi : M \to N$ be a smooth mapping. For a differentiable function $g : N \to \mathbb{R}$, define the *pullback* $\phi^* g : M \to \mathbb{R}$ to be the composition $g \circ \phi$. This gives a linear operator $\phi^* : C^\infty(N) \to C^\infty(M)$, which we extend to differential forms below.

For a vector $X \in TM$, we define $\phi_* X \in TN$ by

$$\phi_* X(g) := X(\phi^* g).$$

This operation, called *pushforward*, gives a vector bundle map $\phi_* : TM \to TN$ which is a lift of $\phi : M \to N$.

We let $\phi_{*x} : T_x M \to T_{\phi(x)} N$ denote the pointwise map of vector spaces. Sometimes we use the notation $d\phi_x$ for ϕ_{*x}. Note that $\phi_{*x} \in T_x^* M \otimes T_{\phi(x)} N$ induces a transpose map, denoted $\phi^* : T_{\phi(x)}^* N \to T_x^* M$, that extends to a bundle map $\phi^* : \Omega^1(N) \to \Omega^1(M)$. More generally, we get induced maps on all tensor constructs (see §A.1), e.g. $\phi_x^* : \Lambda^k T_{\phi(x)}^* N \to \Lambda^k T_x^* M$, which extend to bundle maps. The maps ϕ^* are sometimes called *pullback* maps.

Exercises B.1.6:

1. Show that $\phi^* \alpha(X_1, \ldots, X_k) = \alpha(\phi_* X_1, \ldots, \phi_* X_k)$.
2. Show that $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$.

B.2. Three definitions of the exterior derivative

We now extend the exterior derivative of a function to differential forms of all degrees:

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

Local coordinate definition. Suppose $\alpha = a_{i_1,\ldots,i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ for smooth coefficients $a_{i_1,\ldots,i_k}$. Then

$$d\alpha := \frac{\partial a_{i_1,\ldots,i_k}}{\partial x^l} dx^l \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Note that if $f \in \Omega^0(M) = C^\infty(M)$, then *df* defined this way agrees with our previous definition.

Exercise B.2.1: Show that d is well-defined. (Hint: Use the chain rule to rewrite α and $d\alpha$ in another local coordinate system.)

Vector field definition. If $\alpha \in \Omega^1(M)$ and $X, Y \in \Gamma(TM)$, then

$$d\alpha(X, Y) := X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]). \tag{B.2}$$

More generally, for $\alpha \in \Omega^k(M)$ and $X_1, \ldots, X_{k+1} \in \Gamma(TM)$,

$$\begin{aligned} d\alpha(X_1, \ldots, X_{k+1}) := &\sum_i (-1)^{i+1} X_i \left(\alpha(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})\right) \\ &- \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}), \end{aligned}$$

where $\widehat{\ }$ denotes omission.

Exercises B.2.2:

1. Show that, according to the second definition, $d\alpha$ is $C^\infty(M)$-linear in each X_i. Hence

$$d\alpha(X_1, \ldots, X_{k+1})|_x$$

depends only on the values of the X_i at x.

2. Show that the two definitions of d agree. (Hint: Show that they agree on $(k+1)$-tuples of local coordinate basis vectors.)
3. Using the second definition, show that $\phi^* d\alpha = d(\phi^*\alpha)$.
4. Show that $d^2 = 0$ (i.e., $d(d\alpha) = 0$) under both definitions. In particular, $d^2 = 0$ is equivalent to the fact that mixed partials commute. What property of vector fields is it equivalent to?

Fancy definition. Define d to be the unique operator such that

i. for $f \in C^\infty(M)$, df is as defined in §B.1,
ii. $d(f\alpha) = df \wedge \alpha + f d\alpha$,
iii. $d(\alpha \wedge \beta) = d\alpha \wedge \eta + (-1)^p \alpha \wedge d\beta$ when $\alpha \in \Omega^p(M)$, and
iv. $d^2 = 0$.

Exercise B.2.3: Show that this definition is equivalent to the others.

Geometric interpretation of the exterior derivative of a 1-form. Note that if $\alpha \in \Omega^1(M)$ is nowhere zero, then $\ker \alpha_x$ is a hyperplane in T_xM. Thus, $\ker \alpha$ is a hyperplane field in TM.

If $\alpha = df$, then this hyperplane field is tangent to the level surfaces of f. Otherwise, let E be the quotient bundle $T^*M/\{\alpha\}$. Then $d\alpha \bmod \alpha$ denotes the image of $d\alpha$ under the projection $\Lambda^2 T^*M \to \Lambda^2 E$, and measures how much the hyperplane field defined by α fails to be tangent to hypersurfaces.

Exterior derivatives of vector-valued 1-forms. Let V be a vector space of dimension r. A section of the bundle $\Lambda^k T^*M \otimes V$ is called a *V-valued k-form.* The space of such forms is denoted by $\Omega^k(M, V)$.

Given a basis $v^1, \ldots, v^r \in V$, suppose that $\alpha = \alpha_s \otimes v^s$ for $\alpha_s \in \Omega^k(M)$. Then we extend d for vector-valued forms by defining $d\alpha = d\alpha_s \otimes v^s$. The reader may verify that this is independent of choice of basis.

Definition B.2.4. A form $\alpha \in \Omega^k(M, V)$ is said to be *closed* if $d\alpha = 0$.

The Lie derivative. Let $X \in \Gamma(TM)$ be a vector field. While we already know how to differentiate functions in the direction of X, the *Lie derivative*, denoted by $\mathcal{L}_X$, enables us to differentiate vector fields, differential forms, or sections of any of the tensor product bundles $(T^*M)^{\otimes k} \otimes (TM)^{\otimes l}$.

The Flowbox Theorem 1.2.2 implies that on a sufficiently small neighborhood U of any point in M, there exists a family of local diffeomorphisms $\phi_t : U \to M$ such that $\phi_0 = \mathrm{Id}$ and for any $x \in U$, $t \mapsto \phi_t(x)$ is an integral curve of X. For $\alpha \in \Omega^k(M)$, we define $\mathcal{L}_X\alpha \in \Omega^k(M)$ by

$$\mathcal{L}_X\alpha = \lim_{t\to 0} \frac{1}{t}[\phi_t^*\alpha - \alpha].$$

For $Y \in \Gamma(TM)$, we define $\mathcal{L}_X Y \in \Gamma(TM)$ by

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{1}{t}[(\phi_{-t})_* Y - Y].$$

We extend the Lie derivative to $\Gamma((T^*M)^{\otimes k} \otimes (TM)^{\otimes l})$ by the Leibniz rule. For example, for $\alpha, \beta \in \Omega^1(M)$,

$$\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X\alpha) \otimes \beta + \alpha \otimes \mathcal{L}_X\beta. \tag{B.3}$$

Similarly, $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$.

Exercises B.2.5:

1. Show that $\mathcal{L}_X Y = [X, Y]$.
2. Show that

$$\mathcal{L}_X\alpha = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha), \tag{B.4}$$

where the interior product $\lrcorner$ is as defined in §A.1. (Hint: Prove for 1-forms and extend using wedge products.) This formula for the Lie derivative is due to Cartan.

B.3. Basic and semi-basic forms

Suppose we have a fibration

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ & & \big\downarrow{\scriptstyle \pi} \\ & & B. \end{array}$$

We say that $X \in T_pE$ is a *vertical vector* if $\pi_{*p}X = 0$. Equivalently, X is vertical if $X = i_{*q}Y$ for some $Y \in T_qF$, where $i(q) = p$. A form $\phi \in \Omega^k(E)$ is *semi-basic* if $X \lrcorner\, \phi = 0$ for all vertical vectors X. The form is *basic* if ϕ is a pullback from B, i.e., $\phi = \pi^*(\underline{\phi})$ for some $\underline{\phi} \in \Omega^k(B)$.

Exercise B.3.1: If we let $\{x^i\}$ be local coordinates on B and $\{x^i, y^\alpha\}$ be local coordinates on E, a general 1-form on E may be written as

$$\phi = f_j(x^i, y^\alpha)dx^j + g_\beta(x^i, y^\alpha)dy^\beta.$$

Show that ϕ is semi-basic if $g_\beta \equiv 0$ and basic if, in addition, the f_j are functions of x^i alone.

Remark B.3.2. Similar terminology applies to bundles on E. A bundle $V \to E$ is said to be *basic* if $V = \pi^*(\underline{V})$ for some bundle $\underline{V} \to B$. If V is basic, then a section $s \in \Gamma(V)$ is said to be basic if $s = \pi^*\underline{s}$ for some $\underline{s} \in \Gamma(\underline{V})$.

While it is easy to check if a form is semi-basic, it is not obvious how to check if a form is basic. Fortunately we have the following test:

Proposition B.3.3. *$\alpha \in \Omega^k(E)$ is basic if and only if α and $d\alpha$ are semi-basic.*

Exercise B.3.4: Prove this proposition.

B.4. Differential ideals

Recall that $\Omega^*(M)$ denotes the space of smooth differential forms on a manifold M. This is a graded algebra under the wedge product.

Definition B.4.1. An element in $\Omega^*(M)$ is *homogeneous* if all its terms have the same degree.

Definition B.4.2. A subspace $\mathcal{I} \subset \Omega^*(M)$ is an *algebraic ideal* if it is a direct sum of homogeneous subspaces $\mathcal{I}^k \subset \Omega^k(M)$ and it is closed under wedge product with arbitrary differential forms.

Definition B.4.3. An algebraic ideal $\mathcal{I}$ is a *differential ideal* if it is closed under exterior differentiation, i.e., the exterior derivative of any form in $\mathcal{I}$ is also in $\mathcal{I}$.

Notation B.4.4. If $\omega^1, \dots, \omega^k$ are 1-forms on M, then $\{\omega^1, \dots, \omega^k\} \subseteq \Omega^1(M)$ will denote the linear span of the forms.

Let $\omega^1, \dots, \omega^k$ be forms of arbitrary degree. Then

$$\{\omega^1, \dots, \omega^k\}_{\mathsf{alg}} \subseteq \Omega^*(M)$$

will denote the ideal generated algebraically by $\omega^1, \dots, \omega^n$, i.e., the smallest algebraic ideal containing the generators. Similarly,

$$\{\omega^1, \dots, \omega^k\}_{\mathsf{diff}} \subseteq \Omega^*(M)$$

will denote the ideal generated algebraically by $\omega^1, \dots, \omega^k$ and their exterior derivatives, i.e., the smallest differential ideal containing the generators. For example,

$$\{\omega^1, \omega^2\}_{\mathsf{alg}} = \{\alpha \wedge \omega^1 + \beta \wedge \omega^2 | \, \alpha, \beta \in \Omega^*(M)\},$$
$$\{\omega^1, \omega^2\}_{\mathsf{diff}} = \{\alpha \wedge \omega^1 + \beta \wedge \omega^2 + \gamma \wedge d\omega^1 + \delta \wedge d\omega^2 | \, \alpha, \beta, \gamma, \delta \in \Omega^*(M)\}.$$

Given a subbundle $I \subset T^*M$, we often define a differential ideal $\mathcal{I}$ generated by sections of I. In other words, given a local basis $\omega^1, \ldots, \omega^k$ of sections of I defined on $U \subset M$, then

$$\mathcal{I}|_U = \{\omega^1, \ldots, \omega^k\}_{\mathsf{diff}}.$$

Such a differential ideal is called a *Pfaffian system* of *rank* k (i.e., the rank is the same as that of the vector bundle I).

Complex Structures and Complex Manifolds

Notation. If V is a real vector space, we will let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ (see §A.3). Likewise, if $E \to M$ is a real vector bundle over manifold M, then $E_{\mathbb{C}}$ will denote its complexification. In particular, $T_{\mathbb{C}}M$ is the complexified tangent bundle and $T^*_{\mathbb{C}}M$ the complexified cotangent bundle of M.

C.1. Complex manifolds

In defining a complex manifold, one would like something that is modeled locally on $\mathbb{C}^n$. Two ideas come to mind:

The first idea is to imitate Riemannian manifolds. A Riemannian manifold (M, g) is a space that infinitesimally "looks like" Euclidean space. That is, for all $x \in M$, there exists a positive definite $g_x \in S^2T^*_xM$ that varies smoothly from point to point. One possible definition of a complex manifold would be to mimic this, i.e., define a "complex manifold" to be a real manifold M such that each tangent space T_xM comes equipped with an identification $T_xM \simeq \mathbb{C}^n$ at each $x \in M$, and these identifications vary smoothly from point to point.

The second idea is to imitate the definition of a C^∞ (real) manifold. That is, define a "complex manifold" to be a topological manifold which has local charts into $\mathbb{C}^n$ such that, where the domains of the charts overlap, composing the charts gives invertible holomorphic functions from domains in $\mathbb{C}^n$ to domains in $\mathbb{C}^n$.

Do these two ideas agree? If not, which is the "better" one? The second idea is a local definition, while the first is an infinitesimal one. It's clear that the second idea implies the first, since one can make identifications using the pushforward $\phi_* : T_xM \to T_{\phi(x)}\mathbb{C}^n \simeq \mathbb{C}^n$, and the requirement that composition maps on the overlaps are holomorphic ensures that the identifications are well-defined up to multiplication by a complex matrix. So the question is whether or not the infinitesimal condition is strong enough to imply the local one.

We first need to describe the infinitesimal idea more precisely. Recall from Appendix A that a complex structure on a real vector space V is a map $J \in \mathrm{End}(V)$ with $J \circ J = -\mathrm{Id}$. Recall also that, when we extend J to $V_{\mathbb{C}}$, that space splits as $+i$ and $-i$ eigenspaces of J, denoted $V^{(1,0)} \oplus V^{(0,1)}$ respectively, and that J is uniquely determined by this splitting. So, specifying a complex structure on V is equivalent to specifying the n-dimensional complex subspace $V^{(1,0)} \subset V_{\mathbb{C}}$ (assuming its intersection with its complex conjugate, which will be $V^{(0,1)}$, is zero). In turn, this is equivalent to specifying a dual splitting $V^*_{\mathbb{C}} = V^{*(1,0)}_{\mathbb{C}} \oplus V^{*(0,1)}_{\mathbb{C}}$, where the forms in $V^{*(1,0)}_{\mathbb{C}}$ vanish on the $-i$-eigenspace and vice versa.

Definition C.1.1. An *almost complex structure* on a real $2n$-dimensional manifold M is an n-dimensional smooth distribution $T^{(1,0)}M \subset T_{\mathbb{C}}M$. It determines a unique compatible $J \in \Gamma(\mathrm{End}(TM))$ having $T^{(1,0)}M$ as its $+i$-eigenspace. A manifold equipped with an almost complex structure will be called an *almost complex manifold.*

A *complex structure* on M is an atlas of charts to $\phi_i : U_i \to \mathbb{C}^n$ such that the compositions $\phi_j \circ \phi_i^{-1}$, where defined, are invertible holomorphic functions from domains in $\mathbb{C}^n$ to domains in $\mathbb{C}^n$. A *complex manifold* is a real manifold equipped with a complex structure.

Our question above may be rephrased as, are all almost complex manifolds complex? We will see below that if $\dim M = 2$, the answer is yes.

Proposition C.1.2. *An almost complex structure is complex if and only if $T^{*(1,0)}M \subset \Omega^1_{\mathbb{C}}(M)$ is a Frobenius ideal (equivalently, if $T^{(1,0)}M$ is closed under the Lie bracket). When this is the case, the almost complex structure is said to be* integrable.

Here is the idea of the proof:

At a point p one has a subspace $T^{*(1,0)}_p \subset T^*_{\mathbb{C}}M$ with basis, say, $\alpha^1_p, \ldots, \alpha^n_p$. We want to find local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, such that if we define $z^j = x^j + iy^j$, then $dz^j |_p = \alpha^j_p$ (which we can always do) and moreover we can extend the α^j_p's to sections $\alpha^j \in \Gamma(T^{*(1,0)}M)$ such that in some neighborhood of p we have $dz^j = \alpha^j$. In the real case, this

was exactly the problem we considered in discussing the Frobenius Theorem of Chapter 1, which applies here (with some minor notational adjustments) provided we assume that the underlying manifold is real-analytic and the system $T^{*(1,0)}$ is also analytic. Finally we use $(z^1, \ldots, z^n)$ as our holomorphic coordinate chart. What needs to be checked is that they actually give coordinates which are compatible on the overlaps.

For more discussion of the proof, see [**36**] or [**94**], Chapter IX. The proof in the smooth category is considerably more difficult, and the result in the smooth (in fact C^2) category is known as the Newlander-Nirenberg Theorem.

We will use the notation $\Lambda^{(p,q)}V \subset \Lambda^{p+q}V$ from Appendix A. On an almost complex manifold we have the vector bundles $\Lambda^{(p,q)}T^*_{\mathbb{C}}M$, spanned by wedge products of p $(1,0)$-forms and q $(0,1)$-forms. These are part of a direct sum decomposition

$$\Lambda^k T^*_{\mathbb{C}}M = \bigoplus_{\substack{p+q=k,\\ p\le n, q\le n}} \Lambda^{(p,q)}T^*_{\mathbb{C}}M. \tag{C.1}$$

We refer to smooth sections of $\Lambda^{(p,q)}T^*_{\mathbb{C}}M$ as (p,q)*-forms* and use the notation $\Omega^{(p,q)}(M) = \Gamma(\Lambda^{(p,q)}T^*_{\mathbb{C}}M)$.

A map $f : M \to N$ between almost complex manifolds is called *holomorphic* if df commutes with J, i.e., if for all $x \in M$, $J_{f(x)} \circ f_* = f_* \circ J_x$.

Exercise C.1.3: Show that a function $f : M \to \mathbb{C}$ is holomorphic iff $df \in \Omega^{(1,0)}(M)$.

We let $\Omega^p_{\mathsf{holo}}(M) \subset \Omega^{(p,0)}(M)$ denote the space of *holomorphic p-forms*, which are defined as follows. In local coordinates, $\alpha \in \Omega^{(p,0)}(M)$ may be written $\alpha = a_{i_1,\ldots,i_p} dz^{i_1} \wedge \ldots \wedge dz^{i_p}$ with $a_{i_1,\ldots,i_p} \in C^\infty(M)$. Then α is holomorphic iff all its coefficients $a_{i_1,\ldots,i_p}$ are holomorphic functions.

Exercise C.1.4: Show that $\alpha \in \Omega^{(p,0)}(M)$ is holomorphic iff

$$d\alpha \in \Omega^{(p+1,0)}(M),$$

and moreover, in this case $d\alpha \in \Omega^{p+1}_{\mathsf{holo}}(M)$.

Suppose $\theta_1, \ldots, \theta_n$ give a local framing of $T^{*(1,0)}M$. Then any real $2n$-form on M must be a multiple of the volume form

$$\Omega = i^n \theta_1 \wedge \overline{\theta}_1 \wedge \ldots \wedge \theta_n \wedge \overline{\theta}_n.$$

This form is a nonvanishing section of $\Lambda^{2n}T^*M$ which gives the *canonical orientation* on M associated to the almost complex structure.

Exercise C.1.5: On $\mathbb{R}^{2n}$ with coordinates x^k, y^k for $1 \le k \le n$, let $J(\partial_{x^k}) = \partial_{y^k}$ and $J(\partial_{y^k}) = -\partial_{x^k}$. Show that the corresponding $(1,0)$-forms are spanned by $dz^k = dx^k + i dy^k$, and compute the canonical orientation.

The integrability condition for an almost complex structure is often phrased in terms of its *Nijenhuis tensor*:

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \qquad X,Y \in TM.$$

Through the following exercises, we will see that the vanishing of $N(X,Y)$ for all X, Y is equivalent to the condition in Proposition C.1.2.

Exercises C.1.6:

1. Show that N actually is a tensor, i.e., that N is C^∞-linear in X and Y.
2. Suppose N is extended $\mathbb{C}$-linearly to $T_{\mathbb{C}}M$. Compute N when X and Y are both $+i$-eigenvectors for J, both $-i$-eigenvectors, or one of each.
3. Show that if N is identically zero, then the Lie bracket (suitably extended to vector fields with complex coefficients) of any two $-i$-eigenvectors is another $-i$-eigenvector. Explain why this implies that $T^{*(1,0)}$ satisfies the Frobenius condition.
4. Conversely, show that if $T^{*(1,0)}$ satisfies the Frobenius condition, then the Nijenhuis tensor vanishes.

Examples of complex manifolds.

Example C.1.7 (Almost complex surfaces are complex)**.** Suppose M is a surface. Then there is just one term in the above decomposition (C.1) for $\Lambda^2 T^*_{\mathbb{C}}M$. Thus, $T^{*(1,0)}$ is always integrable. A surface with a complex structure is called a *Riemann surface*.

Since J defines an orientation and $J^2 = -\operatorname{Id}$, then J may be thought of as defining a 90-degree rotation in each tangent space of M. In fact, J defines a metric, up to multiple, at each point. For, suppose θ is a nonvanishing local section of $T^{*(1,0)}$. Then $\Omega = i\theta \wedge \bar{\theta}$, and we may let

$$g(X,Y) = \Omega(X, JY), \qquad X, Y \in TM.$$

Exercise C.1.8: Show that this defines a real-valued, symmetric, positive definite inner product on each tangent space where $\theta \neq 0$.

Of course, changing θ by a nonzero complex multiple changes g by a positive multiple. Thus, a complex structure on a surface is equivalent to a $CO(2)$-structure (see Chapter 8).

Example C.1.9. Let V be an oriented real vector space of dimension n, with inner product $\langle\, , \rangle$. Let $G = Gr(2,V) \cong SO(n)/SO(2) \times SO(n-2)$ be the Grassmannian of oriented 2-planes through the origin in V. Let $E \in G$, and let e^1, e^2 be an oriented orthonormal basis for E^* with respect to the inner product inherited from V. Any vector X in $T_E G$ can be written as

$X = e^1 \otimes v + e^2 \otimes w$ for some $v, w \in V/E$. We define an almost complex structure on G by

$$J(e^1 \otimes v + e^2 \otimes w) = e^1 \otimes w - e^2 \otimes v.$$

Exercise C.1.10: (a) Show that J is well-defined, i.e., it only depends on the inner product and orientation of E, and not on our choice of basis for E^*.
(b) Show that this almost complex structure is integrable. ◎

Almost complex manifolds are far more abundant than complex manifolds and have been used extensively in symplectic geometry—see [**7**]. On the other hand, interesting explicit examples of almost complex structures that are not complex are rare (just as interesting explicit 'generic' Riemannian metrics are rare). Here is one:

Example C.1.11. Let $\mathbb{R}^7 = \mathrm{Im}\mathbb{O}$ with its standard flat metric (see Appendix A for a description of the set $\mathbb{O}$ of octonions). Consider $S^6 \subset \mathrm{Im}\mathbb{O}$. Define $J_u(v) = uv$, where we identify T_uS^6 with the linear subspace $u^\perp \subset \mathbb{R}^7$ and uv is octonionic multiplication.

Exercise C.1.12: Show that this defines an almost complex structure on S^6. Show that the structure is not integrable.

C.2. The Cauchy-Riemann equations

If u, v are respectively the real and imaginary parts of a holomorphic function of $z = x + iy$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{C.2}$$

Since local solutions can be expressed by arbitrary convergent power series in z, we will treat the Cauchy-Riemann equations (C.2) as "solved". For this reason, it is important to recognize systems that could be the Cauchy-Riemann equations in disguise—as in Example 6.4.20. (We may also attempt to find solutions to a given EDS by imposing additional conditions that reduce the system to the Cauchy-Riemann equations [**99**].) To facilitate this, we will examine the structure of the Pfaffian system corresponding to (C.2).

On $J^1(\mathbb{R}^2, \mathbb{R}^2)$ use standard coordinates x^i, u^a, p^a_i (see §1.9), with all indices running between 1 and 2. Let $\Sigma \subset J^1(\mathbb{R}^2, \mathbb{R}^2)$ be the submanifold defined by $p^1_1 = p^2_2$ and $p^1_2 = -p^2_1$. Let $\mathcal{I}$ be the restriction to Σ of the standard contact system generated by forms $\theta^a = du^a - p^a_i\, dx^i$. Then we calculate that

$$d\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} = -\begin{pmatrix}\pi_1 & \pi_2\\ -\pi_2 & \pi_1\end{pmatrix} \wedge \begin{pmatrix}dx^1\\ dx^2\end{pmatrix}, \tag{C.3}$$

where $\pi_1 = dp_1^1$ and $\pi_2 = dp_2^1$. These structure equations can be written more compactly as

$$d\theta = -\pi \wedge \omega, \qquad \text{where } \theta = \theta^1 + i\theta^2,\ \pi = \pi_1 - i\pi_2,\ \omega = dx + i\,dy.$$

The forms θ, ω, π are sections of $T^*_{\mathbb{C}}\Sigma$. The structure equations show that $K = \{\theta, \omega, \pi\}_{\mathbb{C}}$ is the retracting space (see Chapter 6) of the Pfaffian system spanned by θ. The splitting $T^*_{\mathbb{C}} = K \oplus \overline{K}$ defines an almost complex structure on Σ with $K = T^{*(1,0)}\Sigma$. Since K satisfies the Frobenius condition, in this case the almost complex structure is integrable. In this situation, θ is said to define a *complex contact structure* on Σ. A complex analogue of the Pfaff-Darboux Theorem 1.9.17 holds for such forms:

Proposition C.2.1. *Let θ be a smooth nonvanishing complex-valued 1-form on M^n, and suppose $\theta \wedge (d\theta)^m \neq 0$ but $\theta \wedge (d\theta)^{m+1} = 0$ for some m such that $2m + 1 \leq n/2$. Let $K \subset T^*_{\mathbb{C}}M$ be the retracting space of θ, and assume that K is integrable. Then in a neighborhood of any point on M there are complex coordinates w, $z^1, \ldots, z^m$, $p^1, \ldots, p^m$ such that θ is a complex multiple of $dw - p^1 dz^1 \ldots - p^m dz^m$.*

If $m = 1$, then $\theta = dw - p\,dz$ up to multiple, and so integral manifolds of θ may be constructed by setting $w = f(z)$ and $p = f'(z)$ for any holomorphic function $f(z)$.

Remark C.2.2. Note that, in Proposition C.2.1, it is not necessary that M itself carry a complex structure. Instead, this result may be used to locally construct a complex quotient manifold on which the contact structure lives. Of course, it is preferable to identify a global quotient manifold for the problem at hand. For example, the system for superminimal surfaces in S^4 [**16**], which is defined on a ten-dimensional frame bundle, contains a subsystem with structure equations similar to (C.3). Using C.2.1 enables one to locally construct a quotient complex 3-fold on which the subsystem reduces to a holomorphic contact form. However, it is only by identifying a global quotient manifold—$\mathbb{CP}^3$ in the superminimal case—that enables one to obtain results about the space of solutions.

Remark C.2.3. An important application of complex geometry to differential equations is Penrose's twistor program, where, to quote Hitchin [**78**], "the ultimate aim is ... to encode as much of mathematical physics as possible into holomorphic form and there to rely on geometry, supported only by the constraints of the Cauchy-Riemann equations, to provide a description of the universe". See Chapter 8 for an explicit example carried out by Hitchin.

Appendix D

Initial Value Problems

One of the recurring motifs you will find in this book is the attempt to answer the following questions:

> Given a system of differential equations, how large is the space of solutions? What data is sufficient to determine a unique solution? And, given an appropriate set of data, how do we construct this solution?

In this appendix, we review how the classical theory of partial differential equations answers these questions.

Traditionally, an *initial value problem* or *Cauchy problem* consists of a differential equation or system of equations, and *initial data* which specify the value of the unknown function or functions along a codimension-one set H in the domain. The initial value problem is said to be *well-posed* if for all choices of initial data (in some function space) there exists a unique solution to the problem on some open set containing H, or a closed set with H as boundary.

Example D.1. The wave equation $u_{tt} - c^2 u_{xx} = 0$ describes the propagation of a wave in a one-dimensional medium with velocity c, where $u(x,t)$ represents the height of the wave at point x and time t and damping is neglected. Intuitively, since wave motion is the aggregate of the dynamics of many small particles, we expect the solution to be determined by the initial position $u(x,0) = h(x)$ and the velocity $u_t(x,0) = v(x)$. In fact, the general solution of this wave equation is

$$u(x,t) = f(x+ct) + g(x-ct) \tag{D.1}$$

and f, g may be determined from $h(x)$ and $v(x)$. So, the solution is defined for all x and t, and depends on two arbitrary functions of one variable, specified along the hypersurface $t = 0$.

Example D.2. The heat equation $u_t = u_{xx}$ models heat flow in a one-dimensional medium, with $u(x,t)$ representing temperature at point x and time t. Intuitively, one expects that the temperature distribution along $t = 0$ exactly determines the temperature in the domain for all later times, so we are led to pose the problem of solving the heat equation subject to $u = f(x)$ along the hypersurface $t = 0$. In fact, under reasonable assumptions on $f(x)$, there is a unique solution defined on $\mathbb{R} \times [0, \infty)$ given by a convolution formula [**48**].

The usual existence theorems [**85**] for second-order equations modeled on this simple example are focused on producing solutions defined in a neighborhood of H. However, the only *general* existence theorem for systems available to us, namely the *Cauchy-Kowalevski* Theorem stated below, only provides for solutions in a neighborhood of a given point.

While in this sense the Cauchy-Kowalevski Theorem is not very satisfying, it does help illuminate the obvious difference between the size of the solution space for these two second-order equations. The fact that the wave equation solution depends on two functions of x can be explained by recasting the single equation as a system of first-order PDE, and observing that, provided h and v are analytic, the resulting initial value problem fits the hypotheses of Cauchy-Kowalevski. The fact that the heat equation solutions seem to depend on only one function really arises from the fact that the hypersurface $t = 0$, along which the data is specified, is *characteristic* for the heat equation (see §6.2), and as a result Cauchy-Kowalevski cannot be applied to that initial value problem.

Definition D.3. An initial value problem for a system of first-order PDE for s functions u^a of $n+1$ variables is of *Cauchy-Kowalevski form* if it can be written as

$$\frac{\partial u^a}{\partial t} = F^a(x^i, t, u^b, \frac{\partial u^b}{\partial x^i}), \tag{D.2}$$

$$u^a|_{t=t_0} = g^a(x^1, \dots, x^n), \tag{D.3}$$

where $1 \le a, b \le s$ as before, and $1 \le i, j \le n$.

How general is Cauchy-Kowalevski form? If the initial data are specified along some hypersurface, we may always find local coordinates x^i, t so that this becomes a hyperplane where t is constant. However, we then need to be able to solve the equations for the derivatives with respect to this same t.

Restricting to first-order equations is no loss of generality, since any PDE system can be turned into a first-order system by adding extra variables. For example, we can rewrite the wave equation of Example D.1 by letting $u^1 = u$, $u^2 = u_x$ and $u^3 = u_t$. We then obtain the system

$$\begin{cases} \dfrac{\partial u^1}{\partial t} = u^3, \\ \dfrac{\partial u^2}{\partial t} = \dfrac{\partial u^3}{\partial x}, \\ \dfrac{\partial u^3}{\partial t} = c^2 \dfrac{\partial u^2}{\partial x}, \end{cases}$$

with the corresponding initial data $u^1 = h(x)$, $u^2 = h'(x)$ and $u^3 = v(x)$.

Exercises D.4.
1. Give an explicit expression for the solution of the wave equation in terms of initial data $h(x)$ and $v(x)$. Conclude that real-analyticity is not necessary.
2. Show that the initial value problem of Example D.2 cannot be expressed in Cauchy-Kowalevski form. (However, the heat equation by itself can be put in the form (D.2) if we exchange x and t.)

Theorem D.5 (Cauchy-Kowalevski). *Suppose that the system* (D.2) *is quasilinear, i.e.,*

$$F^a(x^i, t, u^b, \frac{\partial u^b}{\partial x^i}) = \sum_{b,i} A^a_{bi}(x, t, u) + B^a(x, t, u)$$

and that the functions A^a_{bi} and B^a are real-analytic on an open set $U \subset \mathbb{R}^{n+1} \times \mathbb{R}^s$. Suppose there is an open set $V \subset \mathbb{R}^n$ on which the initial data (D.3) *is real-analytic, and such that the graph of $g^a|_V$ lies in the intersection of U with the hyperplane $t = t_0$. Then for any point $(z^1, \ldots, z^n) \in V$, there is an open set $W \in \mathbb{R}^{n+1}$ containing $(z^1, \ldots, z^n, t_0)$ and a solution $U^a(x, t)$ of the initial value problem which is real-analytic on W. Moreover, any other solution defined on W must coincide with the U^a's.*

Remark D.6. The restriction of quasilinearity is not essential, since differentiating (D.2) with respect to the x^i produces equations which are linear in the highest-order derivatives. Then adding more variables leads us to a quasilinear system. When generalized to exterior differential systems (EDS), this shows that the prolongation of any EDS is a linear Pfaffian system (see Chapter 5).

Exercise D.7. Add more variables to the nonlinear first-order equation $u\partial u/\partial y = (\partial u/\partial x)^2$ to obtain a quasilinear system.

Why expect the Cauchy-Kowalevski Theorem to be true? Since we are looking for real-analytic solutions, it makes sense to attempt to construct

them via power series in x and t. The special form of (D.2) and (D.3) means that all "mixed" terms—involving x and t—are unambiguously determined in terms of the initial data. For example, consider the simple case of a first-order equation for a single function u of two variables x, t given by

$$\frac{\partial u}{\partial t} = A(x, u)\frac{\partial u}{\partial x} + B(x, u). \tag{D.4}$$

Suppose the initial condition is $u(x, 0) = g(x)$, where A, B and g are analytic functions of their arguments.

Analyticity implies there are convergent power series such that

$$A(x, u) = \sum_{j,s\geq 0} a_{js}x^j u^s, \qquad B(x, u) = \sum_{j,s\geq 0} b_{js}x^j u^s.$$

We want to find an analytic solution

$$u(x, t) = \sum_{j,l\geq 0} c_{jl}x^j t^l$$

for some constants c_{jl}. Since $u(x, 0) = \sum_j c_{j0}x^j$, the initial condition determines the constants c_{j0}. To recover the rest, we substitute in (D.4). To do this, we need the expansions for the powers

$$u^s = \sum_{j,l} c^s_{jl}x^j t^l,$$

where the c^s_{jl} are linear combinations of powers of the c_{jl}. Then (D.4) implies that

$$\sum_{j,l} l c_{jl}x^j t^{l-1} = \sum_{i,j,k,l,m,s} i a_{js}c^s_{kl}c_{im}x^{i+j+k-1}t^{l+m} + \sum_{j,k,l,s} b_{js}c^s_{kl}x^{j+k}t^l.$$

In order for this equality to hold, it must hold for each power of t. Equating the coefficients of the t^0 terms on each side, we get

$$c_{j1}x^j = \sum_{i,j,k,s} i a_{js}c^s_{k0}c_{i0}x^{i+j+k-1} + \sum_{j,k,s} b_{js}c^s_{k0}x^{j+k}.$$

Observe that all terms on the right hand side are known, so if we now equate coefficients of like powers of x, we can solve for the c_{j1}'s. When we equate the coefficients of the t^1 terms, the right hand side will only involve previously known quantities and the c_{j1}'s, while the left hand side will involve the c_{j2}'s. Continuing, we can solve for all the c_{jl}'s.

We conclude that if there is a solution, it must be unique and given by this procedure. To show that a solution in fact exists, it remains to show this series converges in some neighborhood of the origin. For a proof, using the method of majorants, see [**85**]. A special case is given in Chapter 4.

Generalizations. When trying to discover solution formulas for PDE like the d'Alembert formula (D.1), or when studying PDE systems that arise in geometry, we will rewrite the differential equations as exterior differential systems—in other words, replacing coordinates by moving frames, and equations by differential ideals. By appropriately generalizing the notion of characteristic to these systems, we can detect and construct explicit solution formulas for PDE when they exist (see Chapter 6). As well, the Cauchy-Kowalevski Theorem has a generalization which constructs solutions to initial value problems for EDS. This theorem, called the Cartan-Kähler Theorem, is discussed in Chapter 5 and Chapter 7.

Overdetermined Systems. While the Cauchy-Kowalevski Theorem cannot be applied to overdetermined systems (i.e., those with more equations than unknowns), some of these systems do have solutions that depend on more than just constants, and it is possible to solve Cauchy problems for them. However, the equations in such a system must be *compatible*, in the sense that no new higher-order equations result from differentiating the equations in the system and equating mixed partials (see Chapter 1).

In more detail, given a first-order system of r equations for s functions u^a of n variables, there exists a change of coordinates so that the system takes the form

$$\begin{aligned}
u^1_{x^1} &= F^1_1(x^j, u^a),\\
&\vdots\\
u^{r_1}_{x^1} &= F^1_{r_1}(x^j, u^a),\\
u^1_{x^2} &= F^1_2(x^j, u^a, u^a_{x^1}),\\
&\vdots\\
u^{r_2}_{x^2} &= F^{r_2}_2(x^j, u^a, u^a_{x^1}),\\
&\vdots\\
u^1_{x^n} &= F^1_n(x^j, u^a, u^a_{x^1}, \dots, u^a_{x^{n-1}}),\\
&\vdots\\
u^{r_n}_{x^n} &= F^{r_n}_n(x^j, u^a, u^a_{x^1}, \dots, u^a_{x^{n-1}}),
\end{aligned}$$

where $u^a_{x^j} = \frac{\partial u^a}{\partial x^j}$, $1 \le a \le s$, $1 \le j \le n$, and $r_1 \le r_2 \le \ldots \le r_n = s$ with $r = r_1 + \ldots + r_n$ (see [**30**]).

We may be able to produce solutions of this system by solving a series of Cauchy problems. However, we need to check that equations are compatible,

i.e., that mixed partials commute:

$$\frac{\partial}{\partial x^i} F_j^{r_\alpha} = \frac{\partial}{\partial x^j} F_i^{r_\alpha}, \qquad 1 \le i, j \le n, \ \forall r_\alpha.$$

Although it would be impractical to change any given system of PDE into the above form and perform this test, converting this system to an EDS guides us naturally to the analogue of the above form. We can then apply a straightforward test that signals when no further compatibility conditions need to be checked. These topics are the focus of Chapter 4 and Chapter 5.

Hints and Answers to Selected Exercises

Chapter 1

1.4.3(f) Let $A(t)$ be a family of circles with radius $r(t)$ and centers $c(t)$, and show that if $\frac{dr}{dt} > |\frac{dc}{dt}|$ then the circles are nested.

1.7.3.1(d) Consider the area of the region bounded by the curve and a line parallel to the curve, and take the limit as the line approaches the tangent line (see [**67**]).

1.8.4.3 Hint for (b): Write $\bar{c}(t) = c(t) + r(t)e_2(t)$, and show that r is constant.

1.8.4.5 Consider the point $c(t) + \rho N + \sigma B$.

Chapter 2

2.1.1.2 Compare the speed of a curve to its curvature.

2.1.1.3 What is the characteristic polynomial of h?

2.2.2.5 Take a line in the $x - z$ plane and rotate it about the z-axis.

2.2.2.6 Let $r(v) = C\cos(v)$, $t(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v}dv$. The only complete ones are the spheres.

2.3.1.3 Use (B.2).

2.3.1.5 Differentiate your answer from problem 2.3.1.2.

2.3.3 M compact implies there exists a point p at which k_1 has a local maximum. Then K constant implies that k_2 must have a local minimum at p as well. Deduce that p must be an umbilic point. Let $q \in M$ be any other point, and compare $k_j(p)$ with $k_j(q)$.

2.4.1 Answer: $H \equiv 0, K = -\frac{1}{a^2 \cosh^4(v)}$.

2.4.3 Locally we may write $\omega_1^2 = du$ for some function $u : \mathcal{F} \to \mathbb{R}$. Since ω_1^2 is not semi-basic for the projection to M, locally there exists a vertical vector field X such that $\omega_1^2(X) > 0$, or $du(X) > 0$, which implies u is not constant in vertical directions. Thus we may choose a local section $F : M \to \mathcal{F}$ such that $F^*(u)$ is constant and therefore $F^*(\omega_1^2) = 0$.

2.4.5 Instead of calculating the whole frame, just show that $d\omega_1^2 = 0$.

2.5.4.3 It is sufficient to show that if we adapt frames such that e_1 is tangent to the fiber of the Gauss map, then the integral curves to the vector field e_1 are lines; see Chapter 3 for more details.

2.6.4 Hint for (b): differentiate the result of part (a). Hint for (d): use the Cartan Lemma A.1.9.

2.6.9 Show that $\tilde{\Theta}$ is unchanged under the action of $O(n)$ on the fiber of $\mathcal{F}_{\mathsf{on}}(M)$.

2.6.13.4 Differentiating (2.17), with Z in place of X, gives a relationship between Θ and the second derivatives of Z.

2.6.13.3 Hint: differentiate $0 = d\eta^i_j + \eta^i_k \wedge \eta^k_j - \Theta^i_j$.

2.6.17 In indices, the components of $\nabla h = h_{ijk}\eta^i\eta^j\eta^k$ are given by

$$dh_{ij} = h_{ik}\eta^k_j + h_{kj}\eta^k_i + h_{ijk}\eta^k.$$

Then the Codazzi equation amounts to $h_{ijk} = h_{ikj}$.

2.8.2 Use 2.20 and apply Stokes' Theorem to $d\omega_1^2$.

2.9.1 Show that if one refines a triangulation by adding a new vertex and corresponding edges, then χ_Δ doesn't change.

2.10.2 If $\tilde{f} = Af$ for $A \in GL(n)$, then $(g_{ij}(\tilde{f})) = {}^tA^{-1}(g_{ij}(f))A^{-1}$.

2.10.3 First observe that this is true for orthonormal frames.

2.10.4 Normalize $e_1 \times e_2$ to obtain N.

Chapter 3

3.1.1.2 See Appendix A.

3.3.1 Use the fundamental theorem of algebra.

3.3.15.1 When we project, we lose a conormal direction and the tangent space remains the same. Thus $II_{X',x'}$ is the same as $II_{X,x}$, except one quadric is lost. If we let $p = [e_{n+a}]$, and $II_{X,x} = q^{\mu}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\mu}$, then $II_{X,x} = q^{\phi}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\phi}$, where $n+1 \leq \phi \leq n+a-1$.

3.3.15.2 Taking a hyperplane section, we lose a tangent direction; say this direction is $\underline{e}_n$. If $II_{X,x} = q^{\mu}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}\underline{e}_{\mu}$, then $II_{X\cap H,x} = q^{\mu}_{st}\omega^{s}\omega^{t}\underline{e}_{\mu}$, where $1 \leq s, t \leq n-1$.

3.4.4.1 Calculate $d(e_0 + te_1)$.

3.4.4.2 Use Terracini's Lemma.

3.4.4.3 Use 3.4.4.2.

3.5.8.1 Use (3.9).

3.5.8.3 Consider a cubic form in two variables and differentiate.

3.6.1 Use the prolongation property.

3.6.2.3 $T_E G_{\omega}(k, W) \simeq E^* \otimes (E^{\perp}/E) \oplus S^2E^*$. Here $E^{\perp} = \{w \in W \mid \omega(v, w) = 0 \ \forall v \in E\}$.

3.6.5 Note that M_r is a join.

3.6.16.1 Note that $II(w_1 \wedge w_2, u_1 \wedge u_2) = w_1 \wedge w_2 \wedge u_1 \wedge u_2$.

3.9.9 Adapt frames so that $II = \omega^i\omega^j \otimes e_{ij}$ and calculate F_4, F_5. The calculation is a little involved; see [**105**] for the details.

3.10.3 The cases a,b,e are always hypersurfaces, c,d are hypersurfaces respectively when $a = b$ and m is even. The smooth quadric surface in $\mathbb{P}^3$, $v_2(\mathbb{P}^1)$, $G(2,5)$ and $\mathbb{P}^1 \times \mathbb{P}^n$ are all self-dual. The self-duality can be proven directly; or, once one knows the dimension is correct, since the dual variety of a homogeneous variety inherits a group action and they are the dimension of the minimal orbit, they must be the minimal orbit.

3.10.14.1 Given $A \in M_{p\times q}$, consider

$$\begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix}.$$

3.10.14.2 Any nonzero 3×3 skew-symmetric matrix has rank two.

3.10.14.3 Answer: $|II_{\mathrm{Seg}(\mathbb{P}^1\times\mathbb{P}^n)}|$.

3.13.5 Since we are working with $e_{\mu} \operatorname{mod}\{\hat{T}, \widehat{II_v(T)}\}$, adapt frames further such that $\{e_{\xi}\} \simeq II_v(T)$ and $\{e_{\phi}\}$ gives a complement. You will need to show that if $w \in \mathrm{Singloc}(\mathrm{Ann}(v))$, then $II_w(T) \subseteq II_v(T)$.

3.13.6.1 Note that III can be recovered from $III(v,v,v)$ for all $v \in T$.

3.14.4 Use Corollary 3.14.2. What is $\ker II$?

3.16.2 If $x \in X$ is a general point and $III_{X,x} = 0$, then a hyperplane osculating to order two at x contains X.

Chapter 4

4.1.11.1 $A = \{p^1_{11} w_1 \otimes v^1 \circ v^1 + p^1_{12} w_1 \otimes v^1 \circ v^2 + p^1_{22} w_1 \otimes v^2 \circ v^2 + p^2_{11} w_2 \otimes v^1 \circ v^1 + p^2_{12} w_2 \otimes v^1 \circ v^2 + p^2_{22} w_2 \otimes v^2 \circ v^2 \mid p^1_{11} + p^2_{22} = 0\}$ has dimension $5 = 6 - 1$.

Chapter 6

6.1.2.1 Use the fact that $\mathcal{L}_{[\mathsf{v},\mathsf{w}]} = \mathcal{L}_{\mathsf{v}} \circ \mathcal{L}_{\mathsf{w}} - \mathcal{L}_{\mathsf{w}} \circ \mathcal{L}_{\mathsf{v}}$. This identity is easily verified on 0-forms and exact 1-forms, and then follows for all forms from the Leibniz rule (B.3) for Lie derivatives.

6.1.2.2 Suppose $\mathcal{L}_{\mathsf{v}}\psi \in \mathcal{I}$ for $\psi \in \mathcal{I}$. Then use the Leibniz rule to show that $\mathcal{L}_{\mathsf{v}}(\alpha \wedge \psi) \in \mathcal{I}$ for any α, and use (B.4) to get $\mathcal{L}_{\mathsf{v}}(d\psi) \in \mathcal{I}$.

6.1.2.3 The most general v is given by (6.2) for $f = -h_z$ and $g = h + zh_z$, where h is an arbitrary function of x, y, z.

6.1.3 Note that if $\psi \in \mathcal{I}$, then $\phi_t^*(\psi) \in \mathcal{I}$ also.

6.1.6.1 Use (B.4).

6.1.6.2 Use the identity $[\mathsf{v},\mathsf{w}] \lrcorner\, \psi = \mathcal{L}_{\mathsf{v}}(\mathsf{w} \lrcorner\, \psi) - \mathsf{w} \lrcorner\, (\mathcal{L}_{\mathsf{v}}\psi)$, which may be derived in the same way as the formula for $\mathcal{L}_{[\mathsf{v},\mathsf{w}]}$ above.

6.1.6.3 Use the formula $\mathsf{v} \lrcorner\, (\alpha \wedge \psi) = (\mathsf{v} \lrcorner\, \alpha) \wedge \psi + (-1)^{\deg \alpha} \alpha \wedge (\mathsf{v} \lrcorner\, \psi)$, and let ψ be one of the algebraic generators of $\mathcal{I}$.

6.1.14.1 $u(x,y) = \frac{2}{5}\left(x \pm \frac{1}{2}y\right)^2$

6.1.14.2 $u(x,y) = \frac{1}{2}(1 - x^2) \pm y$

6.1.14.3 $u(x,y) = 2x^{3/2}/\sqrt{1 - 27y}$. In general, the solution of $u(s,0) = f(s)$ is $u = f(s) - 2yf'(s)^3$, where s is implicitly defined as a function of x, y by $s = x + 3yf'(s)^2$. If $f(s)$ isn't linear, then there exists an s such that $f''(s) \neq 0$, and hence there exists a y such that $\partial s/\partial x$ is undefined. Then $\partial^2 u/\partial x^2$ is undefined.

6.1.18.1 Use the Pfaff Theorem 1.9.17.

6.1.18.2 Using x, y, z, p, q, s as coordinates, the Cauchy characteristic is given by

$$\mathsf{v} = \frac{\partial}{\partial x} + s^2 \frac{\partial}{\partial y} + (p + qs^2)\frac{\partial}{\partial z} + \frac{2}{3}s^3 \frac{\partial}{\partial p} + 2s\frac{\partial}{\partial q}.$$

6.1.21.2 See Proposition B.3.3.

6.2.8 The hyperbolic system of class zero is generated by $(du - f(v)dx) \wedge dy$ and $(dv - g(u)dy) \wedge dx$.

6.3.3 $I^{(2)}$ is spanned by θ_2.

6.3.13 $q = sx - y \ln s + \int \phi(y)dy + \int \psi(s)/s\, ds$ and $z = xp + qy - xys - \frac{1}{2}(x^2 r - y^2 \ln s) + \int y\phi(y)dy - \frac{1}{2}\int \psi(s)/s^2\, ds$.

6.3.15.2 The class-zero system is integrable if f, g are constants. The class-two system is integrable if f, g are exponential functions.

6.4.4.1 Compute $d\Psi$, noting that $dp, dq \equiv 0$ modulo θ, dx, dy. Then mod out by $d\theta$.

6.4.8 First apply the Pfaff Theorem 1.9.17 to obtain local coordinates in which θ is a multiple of $dz - p\,dx - q\,dy$. Then show that, in these coordinates, the generator 2-form can be taken to have the form (6.19).

6.4.10 $\mathcal{M}_{1,2} = \{\theta, dx \pm dq, dy \pm dp\}$.

6.4.12 If $f_z = 0$, we have a first-order PDE for $q = z_y$. If $f_z \neq 0$, then the equation is semi-integrable if and only if $f(x, y, z, q) = A(x, y, z)q + B(x, y, z)$ and $A_y = B_z$.

6.4.15 For part (b), note that F_r is covered by the transformation on $\mathcal{F}$ given by $(x; e_1, e_2, e_3) \mapsto (x + re_3; e_1, e_2, e_3)$. Then use (1.26) and (1.27) to compute pullbacks of the ω's.

6.4.16.1 Hint: when is $\Theta + \lambda\Psi$ decomposable, modulo ω^3?

6.4.16.2 In a space form of constant curvature ϵ, the Weingarten equation $AK + 2BH + C = 0$ is hyperbolic if and only if $B^2 - A(C + \epsilon A) < 0$. For example, the system for flat surfaces in S^3 is equivalent to the wave equation.

6.4.19 Hint: It's easier to use the ansatz $u = 2\arctan(f(x + y))$.

6.4.23 The minimal surfaces produced in .1 are a plane, Enneper's surface, the catenoid, and the helicoid, respectively. The catenoid is also the mystery surface in .3.

6.5.8 Let $V \subset TB$ be the bundle of vertical vectors for π. Then $J \subset T^*B$ defines a splitting T^*B as the direct sum $J \oplus V^\perp$ of vector bundles, with dual splitting $TB = V \oplus H$.
Let $\theta^i \in \Gamma(J)$ be a (local) basis for sections of J, with dual basis $e_i \in \Gamma(V)$. Then by definition

$$d\theta^i \equiv \Theta^i_j \wedge \theta^j \bmod \pi^*\mathcal{I}$$

for some forms Θ^i_j on B. Thinking of H as a kind of connection, we define the *torsion* of J as

$$\Theta = \Theta^i_j \otimes (\theta^i \otimes e_j) \in \Gamma((T^*B/(J \oplus \pi^*\mathcal{I}^1)) \otimes \mathrm{End}(V)).$$

Then $(B, \mathfrak{J})$ splits locally if there exists a subbundle $W \subset V$ preserved by Θ.

6.5.13.2 Condition (6.36a) implies that decomposable 2-forms in one system are congruent to decomposables for the other, modulo the contact forms $\omega^3, \bar{\omega}^3$. The asymptotic lines are dual to the factors of these decomposables.

6.5.13.5 Although the Cole-Hopf transformation can be described in terms of a double fibration, with the heat equation on one side and Burger's equation on the other, the pullback of the heat equation system doesn't include the 2-forms for Burger's equation. Equivalently, given a solution of the heat equation, there is no Frobenius system available for us to produce multiple solutions of Burger's equation.

6.5.13.6 The point is to show that Bäcklund-equivalence is transitive: given $(M_1, \mathcal{I}_1)$ linked to $(M_2, \mathcal{I}_2)$ by a double fibration carrying a Bäcklund transformation, and $(M_2, \mathcal{I}_2)$ similarly linked to $(M_3, \mathcal{I}_3)$, use this data to construct a Bäcklund transformation linking $(M_1, \mathcal{I}_1)$ and $(M_3, \mathcal{I}_3)$.

Chapter 7

7.1.5 Consider the bundle map $i : T\Sigma \to T^*\Sigma$ given by $\mathsf{v} \mapsto \mathsf{v} \lrcorner (\omega^1 \wedge \omega^2)$, which is well-defined up to multiple. Then the singular points of $\mathcal{V}_2(\mathcal{I})$ are those 2-planes to which this map restricts to be zero.

7.1.7.3 If $\psi \in \mathcal{I}^i$ for $k \leq n$, then $v \lrcorner \psi|_E \neq 0$ implies that there exists $\psi \wedge \alpha \in \mathcal{I}^{n+1}$ such that $v \lrcorner \psi|_E \neq 0$.

7.2.1.1 Express the right-hand side of (7.6) in terms of the Ω^i and solve for the Ω^i in terms of the $d\omega^i \wedge \omega^i$'s.

7.2.1.3 Consider the intersection of a hyperboloid and an ellipsoid in a system of confocal quadrics.

7.5.2 Let $q = \kappa e^{i\theta}$ and differentiate $N = e_2 \sin\theta + e_3 \cos\theta$.

7.5.8.1 In fact, $d(x+(1/k_0)e_3)\wedge\omega^1=0$, so each line of curvature in the e_2-direction lies on a sphere of radius $1/k_0$. The surface is the envelope of these spheres, and the two functions may be taken to be the curvature and torsion of the curve traced out by the centers of the spheres.

7.5.8.2 These surfaces may also be characterized as those for which both of the focal surfaces are degenerate. One such surface is the torus swept out by revolving a circle about an axis. The EDS for these surfaces becomes Frobenius after two prolongations, and the solutions are the *cyclides of Dupin*. These are the surfaces that are obtained by acting on circular tori of revolution by the group $O(4,1)$ of conformal transformations of $\mathbb{E}^3$. See [**84**] for more information, including an interesting application of conformal moving frames.

7.5.8.4 Add the 3-form $\omega_1^4\wedge\omega^2\wedge\omega^3+\omega_2^4\wedge\omega^3\wedge\omega^1+\omega_3^4\wedge\omega^1\wedge\omega^3$ to the ideal.

Chapter 8

8.1.3 All are equivalent.

8.1.12 It is clear that the map $\mathfrak{so}(V)\otimes V^*\to V\otimes V^*\otimes V^*$ is injective. Now show, by switching indices six times, that any tensor in $(\mathfrak{so}(V)\otimes V^*)\cap(V\otimes S^2V^*)$ is zero.

8.5.1 Show that the components of θ span a codimension-one Pfaffian system on $\alpha^*\mathcal{F}_G$.

8.5.2 Use the equivariance of θ.

8.6.10 Consider H^2 as half of a two-sheeted hyperboloid in Lorentz space.

Appendix A

A.1.6.3 Use the preceding exercise.

A.1.8.2 Λ^kV is the span of the wedge products (A.3).

A.1.8.4 $e_{i_1}\wedge\cdots\wedge e_{i_k}$ with $1\le i_1<i_2...<i_k<n$ is a basis of Λ^kV.

A.2.2.1 Consider the kernel of f.

A.2.2.2 Consider the map $\psi: V\otimes S^2V\to\Lambda^2V\otimes V$ given by $w\otimes v\otimes v\mapsto v\text{w}w\otimes v$, and show that the image of ψ is $K_\Lambda(V)$.

A.3.1.1 Consider the characteristic polynomial of J.

A.5.3.4 $\phi=dx^{123}+dx^{345}+dx^{156}+dx^{246}-dx^{147}-dx^{367}+dx^{257}$, where we have written dx^{ijk} for $dx^i\wedge dx^j\wedge dx^k$.

A.6.3 Take a basis of $\mathbb{C}^{d+1}$ consisting of points that project to be on C_d.

A.6.4 See Exercise A.2.3.4.

A.6.5 Schur's Lemma A.2.1 holds for Lie algebras, and the span of an eigenvector gives a subrepresentation.

A.6.16.1 Take traces.

A.7.9.3 Calculate $||u \circ v \circ u||_Q$.

Appendix C

C.1.10(b) Hint: Recall from Chapter 3 that the forms ω_i^k for $3 \leq k \leq n$ and $i = 1, 2$ are semi-basic for the projection from $GL(V)$ to G. Show that the system spanned by the forms $\omega_1^k - i\omega_2^k$ is Frobenius, drops to G and forms a basis for $T^*_{(1,0)}$.

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